Formal Modeling in Cognitive Science 1 (2005–2006)

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Solutions for Tutorial 6: Combinatorics, Basic Probability, Bayes' Theorem

Week 7 (20–24 February 2006)

1. **Combinatorics**

   (a) Assume that all monosyllabic words of English consist of exactly three letters. The first letter can be a consonant or a vowel, the second letter has to be a vowel, and the third letter has to be a consonant. Let the set of vowel be $\{$a, e, i o, u$\}$, the set of consonants the complement of this set.

   i. How many three-letter words are there if a given letter can occur more than once?
   Solution: $26 \cdot 5 \cdot 21 = 2730$

   ii. How many three-letter words are there if a given letter can occur only once?
   Solution:
   First letter is a consonant: $21 \cdot 5 \cdot 20 = 2100$
   First letter is a vowel: $5 \cdot 4 \cdot 21 = 420$
   Total: 2520

   (b) For a psychological experiment, you want to recruit subjects from a class of 20 students. The subjects have to participate in the experiment in pairs.

   i. In how many different ways can you choose 10 ordered pairs of subjects?
   Solution: There are 20! ways of ordering the 20 students, and 10! ways of ordering 10 pairs of subjects. Hence the overall number of combinations is $\frac{20!}{10!}$.

   ii. In how many different ways can you choose 10 unordered pairs of subjects?
   Solution: For each pair, there are $2! = 2$ ways of ordering the elements of the pair, hence for 10 pairs, that's 2^{10} orders. We have to divide the solution of the previous question by this. Hence the overall number of combinations is $\frac{20!}{10! \cdot 2^{10}}$.

2. **Basic Probability**

   (a) Let A and B be two events with $P(A) = 0.59$ and $P(B) = 0.30$ and $P(A \cap B) = 0.21$. Compute the following probabilities:

   i. $P(A \cup B)$
   ii. $P(A \cap \bar{B})$
   iii. $P(\bar{A} \cup \bar{B})$
   iv. $P(\bar{A} \cap \bar{B})$

   Solution:

   i. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.68$
   ii. $P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.38$
   iii. $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 0.79$
   iv. $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 0.32$

   (b) In linguistics, we are often interested in $P(w_{n+1}|w_n)$, the probability that a word w_{n+1} occurs given that the previous word is w_n. ($P(w_{n+1}|w_n)$ is sometimes called a *transitional probability*.)

i. Assume you know the transitional probabilities $P(spotted|the)$, $P(dog|spotted)$, and $P(the)$. What's the probability of the sequence *the spotted dog*?

Solution: The generalized multiplication rule is: $P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$ (see Miller & Miller, p. 33). It follows that $P(the \cap spotted \cap dog) = P(the)P(spotted|the)P(dog|the \cap spotted)$. We can approximate this by assuming that $P(dog|the \cap spotted) = P(dog|spotted)$.

ii. Assume that you know that the word *amok* can follow the words *run*, *running*, and *ran*, which occur with the probabilities $P(run) = 0.5$, $P(running) = 0.25$, and $P(ran) = 0.25$. You also know that the transitional probabilities $P(amok|run) = 0.3$, $P(amok|running) = 0.2$, and $P(amok|ran) = 0.1$. What is the overall probability of seeing *amok*, i.e., $P(amok)$?

Solution: We can apply the rule of total probability: $P(amok) = P(run)P(amok|run) + P(running)P(amok|running) + P(ran)P(amok|ran) = 0.5 \cdot 0.3 + 0.25 \cdot 0.2 + 0.25 \cdot 0.1 = 0.225$.

(c) A balanced die is tossed twice. Let A be the event that an even number comes up on the first toss, B be the event that an even number comes up on the second toss, and C the event that both tosses result in the same number. Which of the events A, B, and C dependent, which ones are independent?

Solution: The events A, B, and C are pairwise independent. $A = \{(2,1),(2,2),\ldots,(4,1),(4,2),\ldots,(6,1),(6,2),\ldots\}$, hence $P(A) = \frac{18}{36} = \frac{1}{2}$ by the rule of equally likely outcomes. In the same way, you can work out $P(B) = \frac{18}{36} = \frac{1}{2}$ and $P(A \cap B) = \frac{9}{36} = \frac{1}{4}$. Hence $P(A)P(B) = P(A \cap B)$, so A and B are independent. The same computation can be used to show the independence of A and C and B and C.

3. **Bayes' Theorem**

Assume that the prevalence of the disease *ritengitis* in the general population 1 in 500. Fever is a symptom of ritengitis, which is present in 3 in 10 cases. Fever is also a symptom of the disease *mesiopathy*, present in 7 in 10 cases. The prevalence of *mesiopathy* is 1 in 100. If a patient presents with fever, what is the probability that they have ritengitis?

Solution: $P(R) = \frac{1}{500}$; $P(F|R) = \frac{3}{10}$; $P(F|M) = \frac{7}{10}$; $P(M) = \frac{1}{100}$

Using Bayes' Theorem: $P(R|F) = \frac{P(R)P(F|R)}{P(R)P(F|R)+P(M)P(F|M)} = \frac{\frac{1}{500} \cdot \frac{3}{10}}{\frac{1}{500} \cdot \frac{3}{10} + \frac{1}{100} \cdot \frac{7}{10}} = \frac{\frac{3}{5000}}{\frac{3}{5000} + \frac{7}{1000}} = \frac{3}{38} =$ 0.0789