Formal Modeling in Cognitive Science

Lecture 22: Expectation and Variance; Chebyshev's Theorem

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1. Expectation and Related Concepts
   - Expectation
   - Mean
   - Variance
2. Chebyshev's Theorem

Expectation

The notion of *mathematical expectation* derives from games of chance. It's the product of the amount a player can win and the probability of wining.

Example

In a raffle, there are 10,000 tickets. The probability of winning is therefore $\frac{1}{10{,}000}$ for each ticket. The prize is worth \$4,800. Hence the expectation per ticket is $\frac{\$4{,}800}{10{,}000} = \$0.48$.

In this example, the expectation can be thought of as the average win per ticket.

Expectation

This intuition can be formalized as the expected value of a random variable:

Definition: Expected Value

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, then the expected value of X is:

$$E(X) = \sum_x x \cdot f(x)$$

We will only deal with the discrete case here (but the definition can be extended to cover continuous random variables).

Expectation

Example

A balanced coin is flipped three times. Let X be the number of heads. Then the probability distribution of X is:

$$f(x) = \begin{cases} \frac{1}{8} & \text{for } x = 0 \\ \frac{3}{8} & \text{for } x = 1 \\ \frac{3}{8} & \text{for } x = 2 \\ \frac{1}{8} & \text{for } x = 3 \end{cases}$$

The expected value of X is:

$$E(X) = \sum_x x \cdot f(x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2}$$

Expectation

The notion of expectation can be generalized to cases in which a function $g(X)$ is applied to a random variable X.

Theorem: Expected Value of a Function

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, then the expected value of $g(X)$ is:

$$E[g(X)] = \sum_x g(x)f(x)$$

Expectation

Example

Let X be the number of points rolled with a balanced die. Find the expected value of X and of $g(X) = 2X^2 + 1$.

The probability distribution for X is $f(x) = \frac{1}{6}$. Therefore:

$$E(X) = \sum_x x \cdot f(x) = \sum_{x=1}^{6} x \cdot \frac{1}{6} = \frac{21}{6}$$

$$E[g(X)] = \sum_x g(x)f(x) = \sum_{x=1}^{6} (2x^2 + 1)\frac{1}{6} = \frac{94}{6}$$

Mean

The expectation of a random variable is also called the *mean* of the random variable. It's denoted by μ.

Definition: Mean

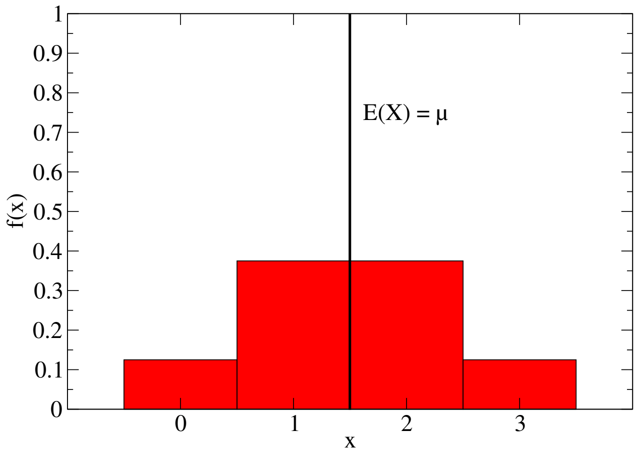
If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, then the mean of X is:

$$\mu = E(X) = \sum_x x \cdot f(x)$$

Intuitively, μ denotes the *average* value of X.

Mean

Histogram with mean for the distribution in the previous example (number of heads in three coin flips):



Variance

Definition: Variance

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, and μ is its mean then:

$$\sigma^2 = \mathrm{var}(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

is the variance of X.

Intuitively, $\mathrm{var}(X)$ reflects the *spread* or *dispersion* of a distribution, i.e., how much it diverges from the mean.

σ is called the *standard deviation* of X.

Variance

Example

Let X be a discrete random variable with the distribution:

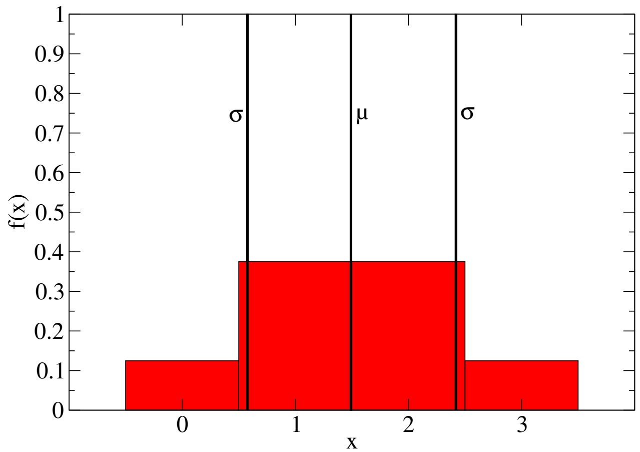
$$f(x) = \begin{cases} \frac{1}{8} & \text{for } x = 0 \\ \frac{3}{8} & \text{for } x = 1 \\ \frac{3}{8} & \text{for } x = 2 \\ \frac{1}{8} & \text{for } x = 3 \end{cases}$$

Then the variance and standard deviation of X are:

$$\begin{aligned} \mathrm{var}(X) &= \sum_x (x-\mu)^2 f(x) \\ &= (0-\frac{3}{2})^2 \cdot \frac{1}{8} + (1-\frac{3}{2})^2 \cdot \frac{3}{8} + (2-\frac{3}{2})^2 \cdot \frac{3}{8} + (3-\frac{3}{2})^2 \cdot \frac{1}{8} \\ &= 0.86 \\ \sigma &= \sqrt{\mathrm{var}(X)} = 0.93 \end{aligned}$$

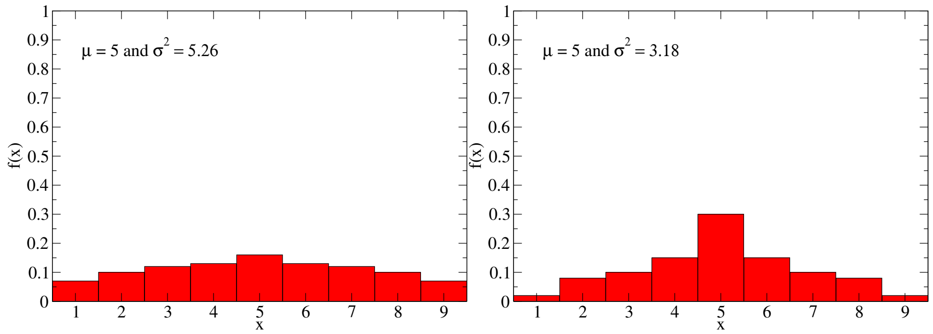
Variance

Histogram with mean and standard deviation for the previous example:



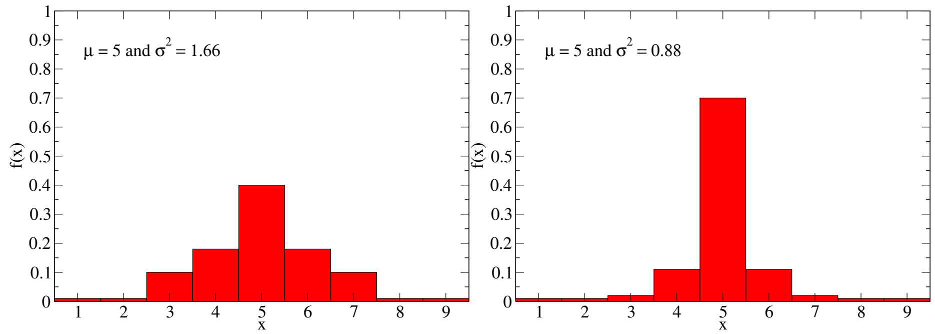
Dispersion

σ^2 as a measure of dispersion:



Dispersion

σ^2 as a measure of dispersion:



Chebyshev's Theorem

Chebyshev's Theorem

If μ and σ are the mean and the standard deviation of a random variable X, and $\sigma \neq 0$, then for any positive constant k:

$$P(|x - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

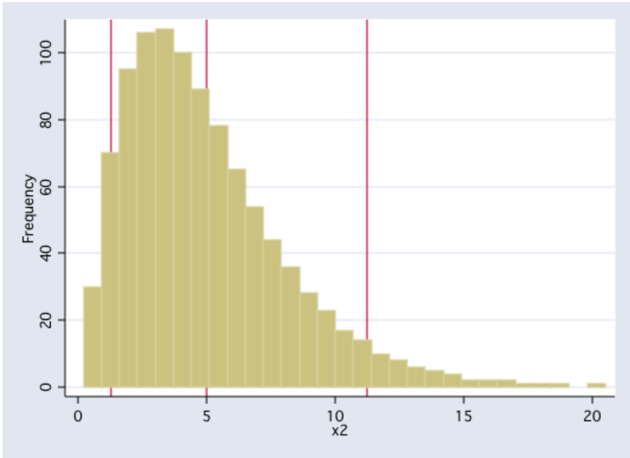
In other words, the probability that X will take on a value within k standard deviations of the mean is at least $1 - \frac{1}{k^2}$.

Example

Assume $k = 2$. Then $P(|x - \mu| < 2\sigma) = 1 - \frac{1}{2^2} = \frac{3}{4}$, i.e., at least 75% of the values of X fall within 2 standard deviations of the mean.

Chebyshev's Theorem

Example: distribution with $\mu = 4.99$ and $\sigma = 3.13$.



Chebyshev's Theorem

Example

Using Chebyshev's Theorem, we can show: if X is normally distributed, then:

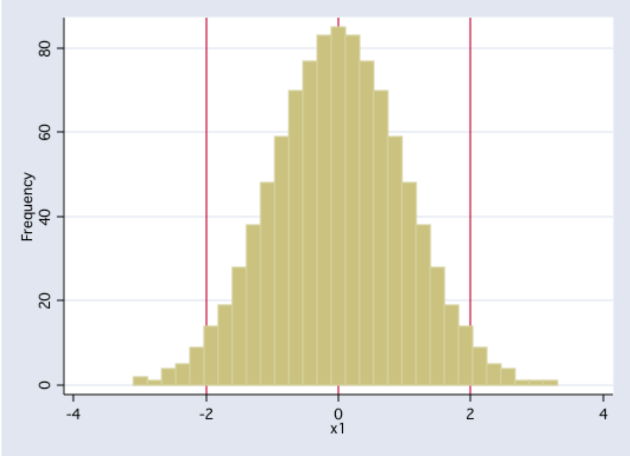
$$P(|x - \mu| < 2\sigma) = .9544$$

In other words, the 95.44% of all values of X fall within 2 standard deviations of the mean. This is a tighter than the bound of 75% that holds for an arbitrary distribution.

Many cognitive variables (e.g., IQ measurements) are normally distributed. More on this in the next lecture.

Chebyshev's Theorem

Example: normal distribution with $\mu = 0$ and $\sigma = 1$.



Summary

- The expected value of a random variable is its average value over a distribution;
- the mean is the same as the expected value;
- the variance of random variable indicates its dispersion, or spread around the mean;
- Chebyshev's theorem places a bound on the probability that the values of a distribution will be within a certain interval around the mean;
- for example, at least 75% of all values of a distribution fall within two standard deviations of the mean.