Formal Modeling in Cognitive Science

Lecture 21: Continuous Random Variables; Densities

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1. Continuous Random Variables

2. Density Functions
   - Probability Density Functions
   - Cumulative Distributions

Continuous Random Variables

We only dealt with discrete (integer-valued) random variables. In many situations, continuous (real-valued) random variables occur.

Examples

The outcomes of real-life experiments are often continuous:

- An experimental subject reacts to a picture by pressing a button (e.g., to indicate if the picture is familiar): the reaction time (in ms) is a continuous random variable.
- An EEG machine measures the electrical brain activity when a subjects reads a word: the current (in μV) is a continuous random variable.

Definition of probability distribution, cumulative distribution, joint distribution, etc., can be extended to the continuous case.

Probability Density Functions

Extend definitions from discrete to continuous random variables:

- use intervals $a \leq X \leq b$ instead of discrete values $X = x$;
- use integration over intervals instead of summation over discrete values.

Definition: Probability Density Function

A function with values $f(x)$, defined over the set of all real numbers, is called a probability density function (pdf) of the continuous random variable X if and only if:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

for any real constants a and b with $a \leq b$.

Probability Density Functions

Example

Assume a continuous random variable X with the pdf:

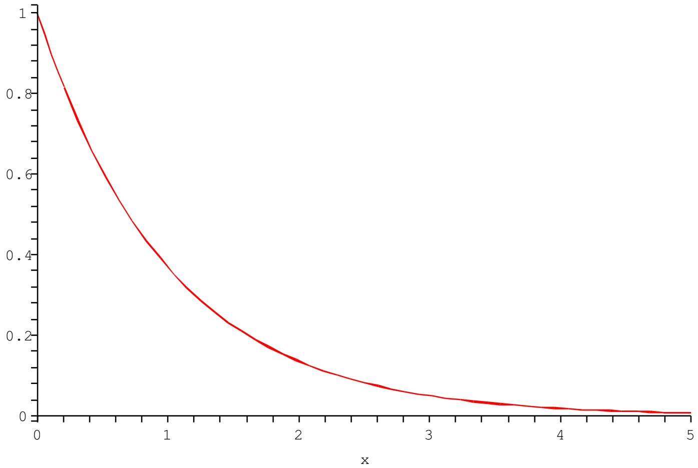
$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Compute the probability for the interval $0 \leq X \leq 1$:

$$\begin{aligned} P(a \leq X \leq b) &= \int_a^b f(x)dx = \int_0^1 e^{-x}dx = -e^{-x}\Big|_0^1 \\ &= (-e^{-1}) - (-e^0) = -\frac{1}{e} + 1 = 0.63 \end{aligned}$$

Probability Density Functions

Plot the function on the previous slide:



Probability Density Functions

Theorem: Intervals of pdfs

If X is a continuous random variable and a and b are real constants with $a \leq b$, then:

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b)$$

Theorem: Valid pdfs

A function can serve as the pdf of a continuous random variable X if its values, $f(x)$, satisfy the conditions:

1. $f(x) \geq 0$ for each value within its domain;
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

Probability Density Functions

Example

Assume a random variable X with the pdf $f(x)$ as follows. Is this a valid pdf?

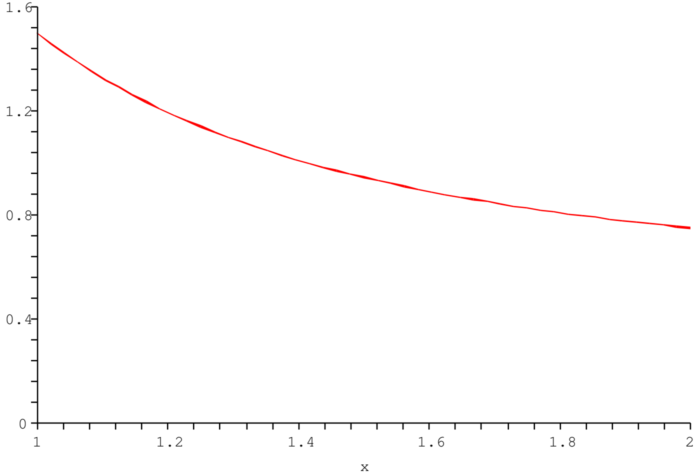
$$f(x) = \begin{cases} \frac{1}{x^2} + \frac{1}{2} & \text{for } 1 < x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

$f(x) \geq 0$ is true by definition. To show $\int_{-\infty}^{\infty} f(x)dx = 1$, integrate:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{1}^{2} \frac{1}{x^2} + \frac{1}{2}dx = -\frac{1}{x} + \frac{1}{2}x \Big|_{1}^{2} \\ &= (-\frac{1}{2} + \frac{1}{2} \cdot 2) - (-\frac{1}{1} + \frac{1}{2} \cdot 1) = 1 \end{aligned}$$

Probability Density Functions

Plot the function on the previous slide:



Cumulative Distributions

In analogy with the discrete case, we can define:

Definition: Cumulative Distribution

If X is a continuous random variable and the value of its probability density function at t is $f(t)$, then the function given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \text{ for } -\infty < x < \infty$$

is the cumulative distribution of X.

Intuitively, the cumulative distribution captures the area under the curve defined by $f(t)$ from $-\infty$ to x.

Cumulative Distributions

Example

Assume a continuous random variable X with the pdf:

$$f(t) = \begin{cases} e^{-t} & \text{for } t > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Integrate for $t > 0$:

$$\begin{aligned} F(x) = P(X \leq x) &= \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} e^{-t}dt = -e^{-t}\Big|_0^x \\ &= (-e^{-x}) - (-e^{0}) = -e^{-x} + 1 \end{aligned}$$

Therefore the cumulative distribution of X is:

$$F(x) = \begin{cases} -e^{-x} + 1 & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Cumulative Distributions

Theorem: Value of Cumulative Distribution

If $f(x)$ and $F(x)$ are the values of the pdf and the distribution function of X at x, then:

$$P(a \leq X \leq b) = F(b) - F(a)$$

for any real constants a and b with $a \leq b$ and:

$$f(x) = \frac{dF(x)}{dx}$$

where the derivative exists.

Cumulative Distributions

Example

Use the theorem on the previous slide to compute the probability $P(0 \leq X \leq 1)$ for $f(t)$:

$$P(0 \leq X \leq 1) = F(1) - F(0) = (-e^{-1}) - (-e^{-0}) = -\frac{1}{e} + 1 = 0.63$$

Also, verify the derivative of $F(x)$:

$$\frac{dF(x)}{dx} = \frac{d(-e^{-x})}{dx} = e^{-x}$$

Other Densities

In analogy with the discrete case, we can define for continuous random variables:

- joint probability density;
- marginal probability density;
- conditional probability density.

Essentially, we replace the $\sum$ signs with integrals in the definitions for the discrete case. We will not deal with this in detail.

Summary

- Probability density functions are the probability distributions for continuous random variables;
- cumulative distributions can also be defined for continuous random variables.