FMCS1: Coursework 3 (Probability) — Solutions

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1. *What is the probability of winning the Lottery?*
   We can describe the situation as follows:
   - *A single play involves choosing 6 different number between 1–49*
   - *The draw chooses 6 numbered balls out of 49*
   - *The order in which the numbers are chosen does not matter in either case*

   (a) *How many possible outcomes of the draw are there?*

   The order of the balls doesn't matter, and so there are $\binom{49}{6}$ possible outcomes.

$$\binom{49}{6} = \frac{49!}{43!6!} = 13983816$$

   (b) *What is the probability of winning the jackpot? (i.e. your 6 chosen numbers match the 6 balls that are drawn.)*

   The probability of winning is

$$\frac{1}{\binom{49}{6}} = \frac{6!43!}{49!} = \frac{6!}{49 \times 48 \times 47 \times 46 \times 45 \times 44} = \frac{1}{13983816}$$

   This makes sense since we are sampling without replacement, and the probability of getting the first ball is is 1/49, of getting the second 1/48, and so on. So the probability of getting all 6 *in the same order that the are drawn* is $1/(49 \times 48 \times 47 \times 46 \times 45 \times 44)$. But any order will do, so we need to multiply by the number of permutations, $6!$.
   We can get to this probability by another route:

$$\frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{3}{46} \times \frac{2}{45} \times \frac{1}{44}$$

   There are 49 balls and 6 match one of the winners initially (6/49). We're sampling without replacement, so this leaves 48 balls, 5 of which are winners, since we have already drawn the 1 winner (5/48). The other terms (4/47, 3/46, 2/45 and 1/44) are added similarly.

(c) *What is the probability of matching at least 5 of the 6 winning balls? (Forget the bonus ball for now.)*

This is given by:

$$\frac{\binom{6}{5}\binom{6}{5}}{\binom{49}{5}} = \frac{6 \times 6 \times 5! \times 44!}{49!} = \frac{6 \times 6!}{49 \times 48 \times 47 \times 46 \times 45} = \frac{1}{52969}$$

In this case the size of the sample space is the number of ways you can choose 5 balls from 49, $\binom{49}{5}$. The number of selections that meet the "5 from 6" criteria is given by $\binom{6}{5}\binom{6}{5}$: the first $\binom{6}{5}$ counts the number of ways you can choose the 5 winning numbers from the 6 balls that were drawn; the second $\binom{6}{5}$ counts the number of ways you can choose the 5 numbers from the 6 numbers that you selected.
Note that this probability is of the form "at least n winning numbers", so it includes the probability of getting more than n winning numbers.

(d) *Now consider the bonus ball (which is a seventh ball that is drawn). What is the probability of matching the bonus ball and 5 of the 6 winning balls?*

This is given by:

$$\frac{\binom{7}{6}}{\binom{49}{6}} = \frac{7 \times 6! \times 43!}{49!} = \frac{7!}{49 \times 48 \times 47 \times 46 \times 45 \times 44} = \frac{7}{13983816} = \frac{1}{1997688}$$

This time the size of the sample space is $\binom{49}{6}$, since 6 winning numbers need to be chosen. Since a winning combination takes 6 numbers, each set of winning numbers corresponds to just one selection of numbers. The number of possible winning selections is given by $\binom{7}{6}$ - the number of 6-number combinations that can be made from the seven drawn numbers (6 + the bonus). Note that this probability includes the winning 6, so the probability of scooping the second prize (and not the jackpot is):

$$\frac{7}{13983816} - \frac{1}{13983816} = \frac{6}{13983816} = \frac{1}{2330636}$$

(e) *To help their readers become millionaires, several newspapers used to publish a "guide" to the lottery listing the numbers that have come up in previous draws, so that people can bet on those numbers that have come up less frequently in the past. Comment on this strategy for choosing lottery numbers.*

Since draws are independent this is an incorrect understanding of the "law of averages". In fact, one would be more justified in using the history to estimate the probability that the draw is loaded in some way (and thus not to bet on infrequently occurring numbers...)

2. *There is a box with three drawers, each containing 2 coins (gold or silver). The contents of the drawers are:*
D1 *2 Gold (GG)*
D2 *1 Gold, 1 Silver (GS)*
D3 *2 Silver (SS)*
You pick a drawer at random, and choose a coin at random from that drawer. The coin is gold. What is the probability that the other coin in the drawer is gold?

We need to calculate $P(C2 = G|C1 = G)$. Using the definition of conditonal probability:

$$P(C2 = G|C1 = G) = \frac{P(C2 = G, C1 = G)}{P(C1 = G)}$$

To calculate $P(C1 = G)$ we need to sum over the three drawers:

$$\begin{aligned} P(C1 = G) &= P(C1 = G, D1) + P(C1 = G, D2) + P(C1 = G, D3) \\ &= P(C1 = G|D1)P(D1) + P(C1 = G|D2)P(D2) + P(C1 = G|D3)P(D3) \\ &= 1 \times 1/3 + 1/2 \times 1/3 + 0 \times 1/3 \\ &= 1/2 \end{aligned}$$

And $P(C2 = G, C1 = G) = 1 \times 1/3 + 0 + 0 = 1/3$ since this is the only two coin combination that can be drawn from D1 (and cannot be drawn from D2 or D3).
So:

$$\begin{aligned} P(C2 = G|C1 = G) &= \frac{P(C2 = G, C1 = G)}{P(C1 = G)} \\ &= \frac{1/3}{1/2} = \frac{2}{3} \end{aligned}$$

3. *There are two identical boxes. Box X contains 600 blue balls and 400 red balls; box Y contains 400 blue balls and 600 red balls. You randomly choose a box (by flipping a fair coin) and draw out 13 balls, replacing the ball after each draw. You obtain 9 blue and 4 red balls. What is the probability that you chose box X?*

Establish notation and write down probabilities (probability estimates) from the question.

- X is chose box X
- Y is chose box Y
- B is drew a blue ball
- R is drew a red ball
- E is the event of drawing 9 blue and 4 red, with replacement

$$P(X) = P(Y) = 0.5 \tag{1}$$
$$P(B|X) = 0.6 = p \tag{2}$$
$$P(R|Y) = 0.6 = p \tag{3}$$
$$P(R|X) = 0.4 = (1-p) \tag{4}$$
$$P(B|Y) = 0.4 = (1-p) \tag{5}$$

Use a binomial to estimate $P(E|X)$ and $P(E|Y)$:

$$P(E|X) = \binom{13}{9} p^9 (1-p)^4 \tag{6}$$
$$P(E|Y) = \binom{13}{9} (1-p)^9 p^4 \tag{7}$$
$$\tag{8}$$
$$P(E) = P(X)P(E|X) + P(Y)P(E|Y) \tag{9}$$
$$= 0.5\binom{13}{9} p^9 (1-p)^4 + 0.5\binom{13}{9} (1-p)^9 p^4 \tag{10}$$
$$= 0.5\binom{13}{9} p^4 (1-p)^4 \left(p^5 + (1-p)^5\right) \tag{11}$$

So use Bayes rule to find $P(X|E)$:

$$P(X|E) = \frac{P(E|X)P(X)}{P(E|X)P(X) + P(E|Y)P(Y)} \tag{12}$$
$$= \frac{\binom{13}{9} p^9 (1-p)^4}{0.5\binom{13}{9} p^4 (1-p)^4 \left(p^5 + (1-p)^5\right)} \tag{13}$$
$$= \frac{p^5}{p^5 + (1-p)^5} \tag{14}$$
$$= \frac{0.6^5}{0.6^5 + 0.4^5} \tag{15}$$
$$= 0.88 \tag{16}$$

88% probability it was box X.