

Elements of Programming Languages

Lecture Notes: L_{Poly}

1 Abstract Syntax

$Expr \ni e ::= n \in \mathbb{N} \mid e_1 + e_2 \mid e_1 \times e_2$	L_{Arith}
$b \in \mathbb{B} \mid e_1 == e_2 \mid \text{if } e \text{ then } e_1 \text{ else } e_2$	L_{If}
$x \mid \text{let } x = e_1 \text{ in } e_2$	L_{Let}
$e_1 e_2 \mid \lambda x : \tau. e$	L_{Lam}
$\text{rec } f(x : \tau_1) : \tau_2. e$	L_{Rec}
$(e_1, e_2) \mid \text{fst } e \mid \text{snd } e \mid \text{let pair } (x, y) = e_1 \text{ in } e_2 \mid ()$	L_{Data}
$\text{left}(e) \mid \text{right}(e) \mid \text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\}$	L_{Data}
$\Lambda A. e \mid e[\tau]$	L_{Poly}
$Type \ni \tau ::= \text{int}$	L_{Arith}
bool	L_{If}
$\tau_1 \rightarrow \tau_2$	L_{Lam}
$\tau_1 \times \tau_2 \mid \text{unit} \mid \tau_1 + \tau_2 \mid \text{empty}$	L_{Data}
$A \mid \forall A. \tau$	L_{Poly}
$\Gamma ::= x_1 : \tau_1, \dots, x_n : \tau_n$	
$Value \ni v ::= n \in \mathbb{N}$	L_{Arith}
$b \in \mathbb{B}$	L_{If}
$\lambda x. e$	L_{Lam}
$\text{rec } f(x). e$	L_{Rec}
$(v_1, v_2) \mid () \mid \text{left}(v) \mid \text{right}(v)$	L_{Data}
$\Lambda A. e$	L_{Poly}

1.1 Fresh variables

In the following, \oplus stands for any binary operator.

$x \# e$

$$\begin{array}{c}
 \frac{n \in \mathbb{N}}{x \# n} \quad \frac{x \# e_1 \quad x \# e_2}{x \# e_1 \oplus e_2} \\
 \frac{b \in \mathbb{B}}{x \# b} \quad \frac{x \# e \quad x \# e_1 \quad x \# e_2}{x \# \text{if } e \text{ then } e_1 \text{ else } e_2} \\
 \frac{x \neq y}{x \# y} \quad \frac{x \# e_2}{x \# \text{let } x = e_1 \text{ in } e_2} \quad \frac{x \neq y \quad x \# e_1 \quad x \# e_2}{x \# \text{let } y = e_1 \text{ in } e_2} \\
 \frac{x \# e_1 \quad x \# e_2}{x \# e_1 e_2} \quad \frac{}{x \# \lambda x:\tau.e} \quad \frac{x \neq y \quad x \# e}{x \# \lambda y:\tau.e} \\
 \frac{}{x \# \text{rec } f(x:\tau) : \tau'.e} \quad \frac{}{f \# \text{rec } f(x:\tau) : \tau'.e} \quad \frac{x \neq f, y \quad x \# e}{x \# \text{rec } f(y:\tau) : \tau'.e} \\
 \frac{x \# e_1 \quad x \# e_2}{x \# (e_1, e_2)} \quad \frac{x \# e}{x \# \text{fst } e} \quad \frac{x \# e}{x \# \text{snd } e} \\
 \frac{x \# e_2}{x \# \text{let } (x, y) = e_1 \text{ in } e_2} \quad \frac{y \# e_2}{y \# \text{let } (x, y) = e_1 \text{ in } e_2} \quad \frac{z \# e_1 \quad z \# e_2}{z \# \text{let } (x, y) = e_1 \text{ in } e_2} \\
 \frac{x \# e}{x \# \text{left}(e)} \quad \frac{x \# e}{x \# \text{right}(e)} \\
 \frac{x \# e \quad x \# e_2}{x \# \text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\}} \quad \frac{y \# e \quad y \# e_1}{y \# \text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\}} \\
 \frac{z \# e \quad z \# e_1 \quad z \# e_2}{z \# \text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\}} \\
 \frac{x \# e}{x \# \Lambda A.e} \quad \frac{x \# e}{x \# e[\tau]}
 \end{array}$$

1.2 Fresh type variables

$A \# \tau$

$$\begin{array}{c}
 \frac{}{A \# \text{int}} \\
 \frac{}{A \# \text{bool}} \\
 \frac{A \# \tau_1 \quad A \# \tau_2}{A \# \tau_1 \times \tau_2} \quad \frac{}{A \# \text{unit}} \quad \frac{A \# \tau_1 \quad A \# \tau_2}{A \# \tau_1 + \tau_2} \quad \frac{}{A \# \text{empty}} \\
 \frac{A \neq B}{A \# B} \quad \frac{}{A \# \forall A.\tau} \quad \frac{A \neq B \quad A \# \tau}{A \# \forall B.\tau}
 \end{array}$$

1.3 Substitution

$$\begin{aligned}
n[e/x] &= n \\
(e_1 \oplus e_2)[e/x] &= e_1[e/x] \oplus e_2[e/x] \\
b[e/x] &= b \\
(\text{if } e_0 \text{ then } e_1 \text{ else } e_2)[e/x] &= \text{if } (e_0[e/x]) \text{ then } (e_1[e/x]) \text{ else } (e_2[e/x]) \\
x[e/x] &= e \\
y[e/x] &= y \quad (x \neq y) \\
(\text{let } y = e_1 \text{ in } e_2)[e/x] &= \text{let } y = e_1[e/x] \text{ in } e_2[e/x] \\
&\quad (\text{where } y \# e) \\
(\lambda y:\tau. e_0)[e/x] &= e_0[e/x] \\
&\quad (\text{where } y \# e) \\
(e_1 e_2)[e/x] &= (e_1[e/x]) (e_2[e/x]) \\
(\text{rec } f(y:\tau):\tau' = e_0)[e/x] &= e_0[e/x] \\
&\quad (\text{where } f, y \# e) \\
(e_1, e_2)[e/x] &= (e_1[e/x], e_2[e/x]) \\
(\text{fst } e_0)[e/x] &= \text{fst } (e_0[e/x]) \\
(\text{snd } e_0)[e/x] &= \text{snd } (e_0[e/x]) \\
(\text{let } (y, z) = e_1 \text{ in } e_2)[e/x] &= \text{let } (y, z) = e_1[e/x] \text{ in } e_2[e/x] \\
&\quad (\text{where } y, z \# e) \\
(\text{left}(e_0))[e/x] &= \text{left}(e_0[e/x]) \\
(\text{right}(e_0))[e/x] &= \text{right}(e_0[e/x]) \\
\text{case } e_0 \text{ of } \{\text{left}(y) \Rightarrow e_1 ; \text{right}(z) \Rightarrow e_2\}[e/x] &= \text{case } (e_0[e/x]) \text{ of } \{\text{left}(y) \Rightarrow e_1[e/x] ; \text{right}(z) \Rightarrow e_2[e/x]\} \\
&\quad (\text{where } y, z \# e) \\
(\Lambda A. e_0)[e/x] &= \Lambda A. (e_0[e/x]) \\
(e_0[\tau])[e/x] &= (e_0[e/x])[\tau]
\end{aligned}$$

2 Evaluation

$e \Downarrow v$ for L_{Arith}

$$\frac{}{v \Downarrow v} \quad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 + e_2 \Downarrow v_1 +_{\mathbb{N}} v_2} \quad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{e_1 \times e_2 \Downarrow v_1 \times_{\mathbb{N}} v_2}$$

$e \Downarrow v$ for L_{If}

$$\frac{e_1 \Downarrow v \quad e_2 \Downarrow v}{e_1 == e_2 \Downarrow \text{true}} \quad \frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \neq v_2}{e_1 == e_2 \Downarrow \text{false}}$$

$$\frac{e \Downarrow \text{true} \quad e_1 \Downarrow v_1}{\text{if } e \text{ then } e_1 \text{ else } e_2 \Downarrow v_1} \quad \frac{e \Downarrow \text{false} \quad e_2 \Downarrow v_2}{\text{if } e \text{ then } e_1 \text{ else } e_2 \Downarrow v_2}$$

$e \Downarrow v$ for L_{Let}

$$\frac{e_1 \Downarrow v_1 \quad e_2[v_1/x] \Downarrow v_2}{\text{let } x = e_1 \text{ in } e_2 \Downarrow v_2}$$

$e \Downarrow v$ for L_{Lam}

$$\frac{}{\lambda x. e \Downarrow \lambda x. e} \quad \frac{e_1 \Downarrow \lambda x. e \quad e_2 \Downarrow v_2 \quad e[v_2/x] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$e \Downarrow v$ for L_{Rec}

$$\frac{}{\text{rec } f(x). e \Downarrow \text{rec } f(x). e} \quad \frac{e_1 \Downarrow \text{rec } f(x). e \quad e_2 \Downarrow v_2 \quad e[\text{rec } f(x). e/f, v_2/x] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$e \Downarrow v$ for L_{Data}

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)} \quad \frac{e \Downarrow (v_1, v_2)}{\text{fst } e \Downarrow v_1} \quad \frac{e \Downarrow (v_1, v_2)}{\text{snd } e \Downarrow v_2} \quad \frac{e_1 \Downarrow (v_1, v_2) \quad e_2[v_1/x, v_2/y] \Downarrow v}{\text{let pair } (x, y) = e_1 \text{ in } e_2 \Downarrow v}$$

$$\frac{e \Downarrow v}{\text{left}(e) \Downarrow \text{left}(v)} \quad \frac{e \Downarrow v}{\text{right}(e) \Downarrow \text{right}(v)}$$

$$\frac{e \Downarrow \text{left}(v_1) \quad e_1[v_1/x] \Downarrow v}{\text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\} \Downarrow v} \quad \frac{e \Downarrow \text{right}(v_2) \quad e_2[v_2/y] \Downarrow v}{\text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\} \Downarrow v}$$

$e \Downarrow v$ for L_{Poly}

$$\frac{}{\Lambda A. e \Downarrow \Lambda A. e} \quad \frac{e \Downarrow \Lambda A. e_0 \quad e_0[\tau/A] \Downarrow v}{e[\tau] \Downarrow v}$$

3 Types

$\boxed{\Gamma \vdash e : \tau}$ for L_{Arith}

$$\frac{}{\Gamma \vdash n : \text{int}} \quad \frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} \quad \frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 \times e_2 : \text{int}}$$

$\boxed{\Gamma \vdash e : \tau}$ for L_{If}

$$\frac{}{\Gamma \vdash b : \text{bool}} \quad \frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 == e_2 : \text{bool}} \quad \frac{\Gamma \vdash e : \text{bool} \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau}$$

$\boxed{\Gamma \vdash e : \tau}$ for L_{Let}

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$ for L_{Lam}

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1 e_2 : \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$ for L_{Rec}

$$\frac{\Gamma, f : \tau_1 \rightarrow \tau_2, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \text{rec } f(x : \tau_1) : \tau_2. e : \tau_1 \rightarrow \tau_2}$$

$\boxed{\Gamma \vdash e : \tau}$ for L_{Data}

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{fst } e : \tau_1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{snd } e : \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_1 \times \tau_2 \quad \Gamma, x : \tau_1, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \text{let pair } (x, y) = e_1 \text{ in } e_2 : \tau}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{left}(e) : \tau_1 + \tau_2} \quad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{right}(e) : \tau_1 + \tau_2} \quad \frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_1 : \tau \quad \Gamma, y : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \text{case } e \text{ of } \{\text{left}(x) \Rightarrow e_1 ; \text{right}(y) \Rightarrow e_2\} : \tau}$$

$\boxed{\Gamma \vdash e : \tau}$ for L_{Poly}

$$\frac{\Gamma \vdash e : \tau \quad A \# \Gamma}{\Gamma \vdash \Lambda A. e : \forall A. \tau} \quad \frac{\Gamma \vdash e : \forall A. \tau}{\Gamma \vdash e[\tau_0] : \tau[\tau_0/A]}$$