Recap

- Given word counts we can estimate a probability distribution:
  \[ P(w) = \frac{\text{count}(w)}{\sum_{w'} \text{count}(w')} \]

- Another useful concept is conditional probability
  \[ p(w_2|w_1) \]

- Chain rule:
  \[ p(w_1, w_2) = p(w_1) \cdot p(w_2|w_1) \]

- Bayes rule:
  \[ p(x|y) = \frac{p(y|x) \cdot p(x)}{p(y)} \]
Expectation

- We introduced the concept of a random variable $X$
  
  $\text{prob}(X = x) = p(x)$

- Example: Roll of a dice. There is a $\frac{1}{6}$ chance that it will be 1, 2, 3, 4, 5, or 6.

- We define the expectation $E(X)$ of a random variable as:
  
  $E(X) = \sum_x p(x) x$

- Roll of a dice:
  
  $E(X) = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 3.5$

Variance

- **Variance** is defined as
  
  $\text{Var}(X) = E((X - E(X))^2) = E(X^2) - E^2(X)$
  
  $\text{Var}(X) = \sum_x p(x) (x - E(X))^2$

- Intuitively, this is a measure how far events diverge from the mean (expectation)

- Related to this is **standard deviation**, denoted as $\sigma$.
  
  $\text{Var}(X) = \sigma^2$
  
  $E(X) = \mu$
Variance (2)

• Roll of a dice:

\[
Var(X) = \frac{1}{6}(1 - 3.5)^2 + \frac{1}{6}(2 - 3.5)^2 + \frac{1}{6}(3 - 3.5)^2 \\
+ \frac{1}{6}(4 - 3.5)^2 + \frac{1}{6}(5 - 3.5)^2 + \frac{1}{6}(6 - 3.5)^2 \\
= \frac{1}{6}((-2.5)^2 + (-1.5)^2 + (-0.5)^2 + 0.5^2 + 1.5^2 + 2.5^2) \\
= \frac{1}{6}(6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25) \\
= 2.917
\]

Standard distributions

• Uniform: all events equally likely
  – \( \forall x, y : p(x) = p(y) \)
  – example: roll of one dice

• Binomial: a series of trials with only only two outcomes
  – probability \( p \) for each trial, occurrence \( r \) out of \( n \) times:
    \[
b(r; n, p) = \binom{n}{r} p^r (1 - p)^{n-r}
\]
  – a number of coin tosses
Standard distributions (2)

- **Normal**: common distribution for continuous values
  - value in the range $[-\infty, x]$, given expectation $\mu$ and standard deviation $\sigma$:
    \[
    n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
    \]
  - also called **Bell curve**, or **Gaussian**
  - examples: heights of people, IQ of people, tree heights, ...

Estimation revisited

- We introduced last lecture an estimation of probabilities based on frequencies:
  \[
  P(w) = \frac{\text{count}(w)}{\sum_{w'} \text{count}(w')}
  \]
- Alternative view: Bayesian: what is the most likely model given the data
  \[
  p(M|D)
  \]
- Model and data are viewed as random variables
  - model $M$ as random variable
  - data $D$ as random variable
Bayesian estimation

- Reformulation of $p(M|D)$ using Bayes rule:
  
  $$p(M|D) = \frac{p(D|M)p(M)}{p(D)}$$

  $$\arg\max_M p(M|D) = \arg\max_M p(D|M) p(M)$$

- $p(M|D)$ answers the question: What is the most likely model given the data

- $p(M)$ is a prior that prefers certain models (e.g. simple models)

- The frequentist estimation of word probabilities $p(w)$ is the same as Bayesian estimation with a uniform prior (no bias towards a specific model), hence it is also called the maximum likelihood estimation

Entropy

- An important concept is entropy:
  
  $$H(X) = \sum_x -p(x) \log_2 p(x)$$

- A measure for the degree of disorder
Entropy example

One event

\[ p(a) = 1 \]
\[ H(X) = -1 \log_2 1 \]
\[ = 0 \]

Entropy example

2 equally likely events:

\[ p(a) = 0.5 \]
\[ p(b) = 0.5 \]
\[ H(X) = -0.5 \log_2 0.5 - 0.5 \log_2 0.5 \]
\[ = - \log_2 0.5 \]
\[ = 1 \]
Entropy example

4 equally likely events:

\[ p(a) = 0.25 \]
\[ p(b) = 0.25 \]
\[ p(c) = 0.25 \]
\[ p(d) = 0.25 \]

\[ H(X) = -0.25 \log_2 0.25 - 0.25 \log_2 0.25 \]
\[ = -0.25 \log_2 0.25 - 0.25 \log_2 0.25 \]
\[ = - \log_2 0.25 \]
\[ = 2 \]

Entropy example

4 equally likely events, one more likely than the others:

\[ p(a) = 0.7 \]
\[ p(b) = 0.1 \]
\[ p(c) = 0.1 \]
\[ p(d) = 0.1 \]

\[ H(X) = -0.7 \log_2 0.7 - 0.1 \log_2 0.1 \]
\[ = -0.7 \log_2 0.7 - 0.1 \log_2 0.1 \]
\[ = -0.7 \times -0.5146 - 0.1 \times -3.3219 \]
\[ = 0.36020 + 0.99658 \]
\[ = 1.35678 \]
Entropy example

4 equally likely events, one much more likely than the others:

\[(X)\]

\[\begin{align*}
p(a) &= 0.97 \\
p(b) &= 0.01 \\
p(c) &= 0.01 \\
p(d) &= 0.01
\end{align*}\]

\[H(X) = -0.97 \log_2 0.97 - 0.01 \log_2 0.01\]

\[= -0.97 \log_2 0.97 - 0.03 \log_2 0.01\]

\[= -0.97 \times -0.04394 - 0.03 \times -6.6439\]

\[= 0.04262 + 0.19932\]

\[= 0.24194\]

Intuition behind entropy

- A good model has low entropy

→ it is more certain about outcomes

- For instance a translation table

| e  | f  | \(p(e|f)\) |
|----|----|------------|
| the | der | 0.8        |
| that| der | 0.2        |

is better than

| e  | f  | \(p(e|f)\) |
|----|----|------------|
| the| der | 0.02       |
| that| der | 0.01       |
| ...| ...| ...        |

- A lot of statistical estimation is about reducing entropy
Information theory and entropy

- Assume that we want to encode a sequence of events $X$

- Each event is encoded by a sequence of bits

- For example
  - Coin flip: heads = 0, tails = 1
  - 4 equally likely events: $a = 00$, $b = 01$, $c = 10$, $d = 11$
  - 3 events, one more likely than others: $a = 0$, $b = 10$, $c = 11$
  - Morse code: $e$ has shorter code than $q$

- Average number of bits needed to encode $X \geq$ entropy of $X$

The entropy of English

- We already talked about the probability of a word $p(w)$

- But words come in sequence. Given a number of words in a text, can we guess the next word $p(w_n|w_1, ..., w_{n-1})$?

- Example: Newspaper article
Entropy for letter sequences

Assuming a model with a limited window size

<table>
<thead>
<tr>
<th>Model</th>
<th>Entropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0th order</td>
<td>4.76</td>
</tr>
<tr>
<td>1st order</td>
<td>4.03</td>
</tr>
<tr>
<td>2nd order</td>
<td>2.8</td>
</tr>
<tr>
<td>human, unlimited</td>
<td>1.3</td>
</tr>
</tbody>
</table>