Distributed Systems

Minimum spanning trees

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Minimum spanning trees

• Definition (in an undirected graph):
  – A spanning tree that has the smallest possible total weight of edges

Ref: Wiki
Minimum spanning trees

- Useful in broadcast:
  - Using a flood on the MST has the smallest possible cost on the network
Minimum spanning trees

• Useful in point to point routing:
  – Minimizes the max weight on the path between any two nodes
Property: Cut optimality

- Every edge of the MST partitions the graph into two disjoint sets (creates a cut)
  - Each set is individually connected by MST edges
Property: Cut optimality

- Every edge of the MST partitions the graph into two disjoint sets (creates a *cut*)
  - Each set is individually connected by MST edges
- No edge across the cut can have a smaller weight than the MST edge
Property: Cut optimality

• Every edge of the MST partitions the graph into two disjoint sets (creates a cut)
  – Each set is individually connected by MST edges
• No edge across the cut can have a smaller weight than the MST edge
• Proof: If there was such an edge, then we can swap it for the current edge and get a tree of smaller total weight
Property: Cycle optimality

- Every non-MST edge when added to MST set creates a cycle
- It must have max weight in the cycle
MST: Not necessarily unique

• Why?
MST: Not necessarily unique

• Assume:
  – All edge weights are unique
Prim’s Algorithm

• Initialize $P = \{x\}; \ Q = E$
  – (x is any vertex in $V$)

• While $P \neq V$
  – Select edge $(u,v)$ in the cut $(P, V\setminus P)$
    • (at the boundary of $P$)
    • With smallest weight
  – Add $v$ to $P$
Prim’s Algorithm

• If we search for the min weight edge each time: $O(n^2)$
Prim’s Algorithm

• If we use *heaps*:
  – $O(m \log n)$ [binary heap]
  – $O(m + n \log n)$ [Fibonacci heap]
Prim’s Algorithm

• Can we have an efficient distributed implementation?
Prim’s Algorithm

• In every round, we need to find the lowest weight boundary edge.

• Use a convergecast (aggregation tree based)
  – In every round
  – For n rounds
Prim’s Algorithm

• What is the running time?
• What is communication complexity?
Prim’s Algorithm

• The weakness:
• Does not use the distributed computation
• Tree spreads from one point, rest of network is idle
Kruskal’s algorithm

• Works with a forest: A collection of trees
• Initially: each node is its own tree
• Sort all edges by weight
• For each tree,
  – Find the least weight boundary edge
  – Add it to the set of edges: merges two trees into one
  – Repeat until only 1 tree left
Kruskal’s algorithm

• The problem step:
  – “Find the least weight boundary edge”
• How do you know which is the boundary edge?
• Maintain id for each tree (store this at every node)
• Easy to check if end-point belong to different trees
• When merging trees, update the id of one of the trees
  – Expensive, since all nodes in the tree have to be updated
Kruskal’s algorithm

• When merging trees, update the id of one of the trees
  – Expensive, since all nodes in the tree have to be updated
• Solution: always update the id of the smaller tree (the one with fewer nodes)
• The cost for all id updates is $O(n \log n)$
Kruskal’s algorithm

• Claim: The cost for all id updates is $O(n \log n)$
• Proof: (by induction on levels)
  – Suppose the final list of $n$ elements was obtained by merging two lists of $h$ elements and $n-h$ elements in the previous level
  – And $h \leq n/2$
  – Then cost of creating final list is (for some const $p$):
    • Cost for creating two lists $\leq ph \log h + p(n-h)\log (n-h)$
    • Cost for updating labels $\leq ph$
    • Total $\leq ph \log h + p(n-h)\log (n-h) + ph$
    • Total $\leq ph (\log (n/2) + 1) + p(n-h)\log (n-h)$
    • $\leq pn \log n$

• Note: Kruskal also needs time to sort the edges initially
GHS Distributed MST Algorithm

• By Gallagher, Humblet and Spira
• Each node knows its own edges and weights

Ref: NL
GHS Distributed MST Algorithm

- Works in levels
- In level 0 each node is its own tree
- Each tree has a leader (leader id == tree id)
- At each level k:
  - All Leaders execute a convergecast to find the min weight boundary edge in its tree
  - It then broadcasts this in its tree so that the node that has the edge knows
  - This node informs the node on the other side, which informs its own leader
GHS Distributed MST Algorithm

• Observation 1:
  – We are possibly merging more than two trees at the same time
  – Problem: who is the leader of the new tree?

• Observation 2:
  – The merged tree is a tree of trees: it cannot have a cycle
  – We can assign a direction to each edge and each node (tree) has an outgoing edge
  – There must be a pair of nodes (trees) that select each-other (otherwise the merged tree is infinite)
  – We select the edge used to merge these two trees
    • Select the node with higher ID to be leader
  – The leader then broadcasts a message updating leader id at all nodes.
GHS Distributed MST Algorithm

• Complexity:
  • The number of nodes at each level $k$ tree is at least $2^k$
  • Since starting at size 1, the number of nodes in the smallest tree at least doubles every level
  • Therefore, there are at most $O(\log n)$ levels
GHS Distributed MST Algorithm

- Complexity:
- At each level, at each tree, we use constant number of broadcasts and convergecasts
- Each level costs $O(n)$ time
- Total costs : $O(n \log n)$ time
GHS Distributed MST Algorithm

• Complexity:
  • At each level, at each tree, we use constant number of broadcasts and convergecasts
  • Each level costs $O(n)$ messages
  • Total costs : $O(n \log n + |E|)$ messages
Distributed MST Algorithm

• Non-unique edge weights
• If edges have duplicate weights
• We make them unique:
  – By ensuring that for any two edges e and e’
  – Either \( wt(e) < wt(e’) \) or \( wt(e’) < wt(e) \)
  – By using node ids
  – Eg. If \((u,v)\) and \((u’,v’)\) have same weight, we define
    • If \( u < u’ \) then \( wt(u,v) < wt(u’v’) \)
    • Else if \( u == u’ \), and if \( v < v’ \) then \( wt(u,v) < wt(u’v’) \)