Decision Making

in Robots and Autonomous Agents

Stochastic System Models: How should a robot reason about uncertainty?

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Lecture Objectives

This lecture has three objectives:

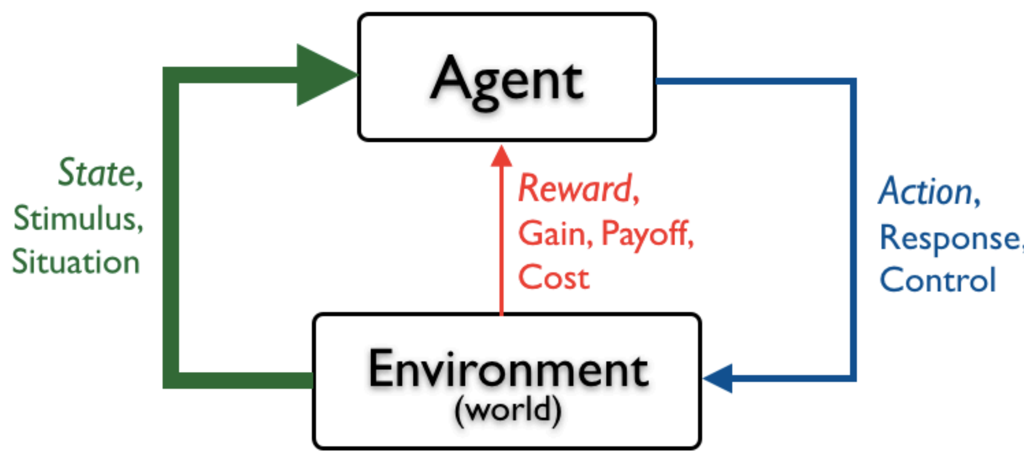
- Introduce you to the notion of reasoning about the distribution of payoffs – using the simplest example of a decision problem, the Multi-armed bandit (MAB)
- Extend this solution concept to address the computation of an optimal policy for the MDP (generalizing what we saw two lectures ago) – key concept being the Bellman equation
- Introduce the computational procedures of Value and Policy Iteration, along with a simple example

Remark: If you are also registered in RL, this lecture *will* have overlap of content

An Agent-Environment View of Robots

Agent (e.g., reinforcement learning algorithm) is:

- Temporally situated
- Continual learning and planning
- Objective is to ***affect*** the environment – actions *and* states
- Environment is uncertain, stochastic



Multi-arm Bandits (MAB)

- *N* possible actions
- You can play for some period of time and *<u>you want to maximize reward</u>* (expected utility)

Which is the best arm/ machine?

DEMO

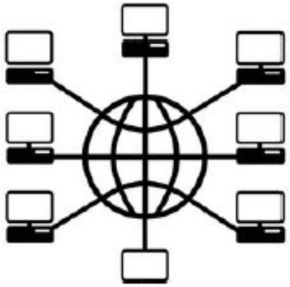
Numerous Applications!

Computer Go

Brain computer interface

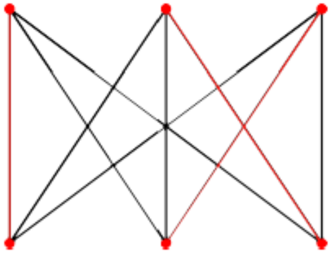
Medical trials

Packets routing

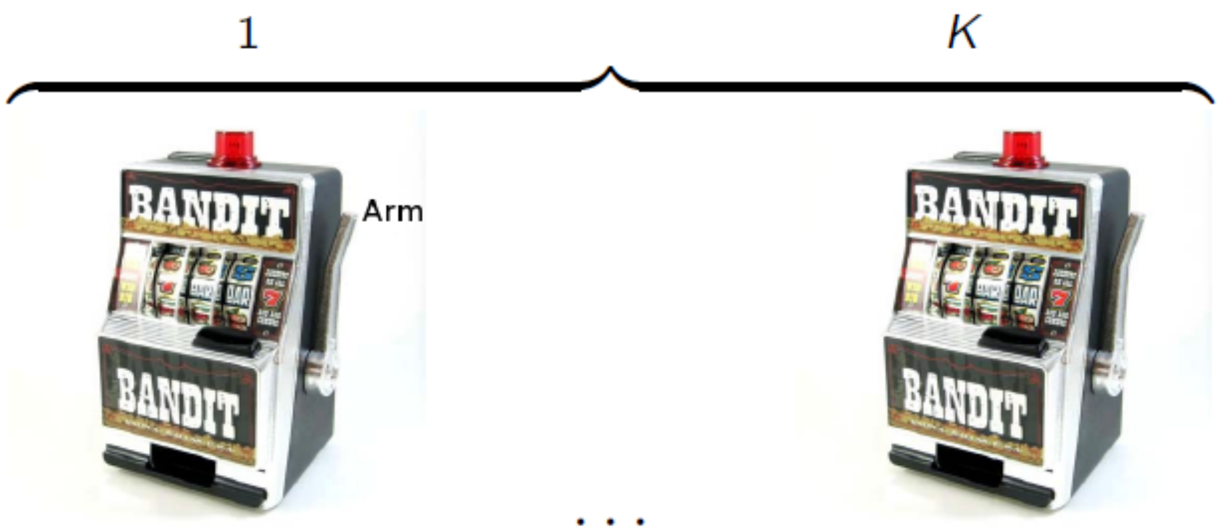


Ads placement

Dynamic allocation



What is the Choice?



t=1	0.3	0.2	0.8	0.4	0.0
t=2	0.7	0.1	0.9	0.5	0.1
t=3	0.5	0.3	0.7	0.3	0.3

...

The n-armed Bandit Problem

- Choose repeatedly from one of *n* actions; each choice is called a *play*
- After each play a_t , you get a reward r_t , where

$$E\{r_t \mid a_t\} = Q^*(a_t)$$

These are unknown ***action values***
Distribution of r_t depends only on a_t

Question: How does this relate to the f (cost to go) in earlier DP lecture?

Objective is to maximize the reward in the long term, e.g., over 1000 plays

To solve the *n*-armed bandit problem,
you must **explore** a variety of actions
and **exploit** the best of them

Exploration/Exploitation Dilemma

- Suppose you form estimates

$$Q_t(a) \approx Q^*(a)$$ action value estimates

- The **greedy** action at time t is a_t^*

$$a_t^* = \arg\max_a Q_t(a)$$

$$a_t = a_t^* \Rightarrow \text{exploitation}$$

$$a_t \neq a_t^* \Rightarrow \text{exploration}$$

Why?

- You can't exploit all the time; you can't explore all the time
- You can never stop exploring; but you could reduce exploring.

Action-Value Methods

- Methods that adapt action-value estimates and nothing else, e.g.: suppose by the t-th play, action a had been chosen k_a times, producing rewards r_1, r_2, …, r_{k_a}, then

$$Q_t(a) = \frac{r_1 + r_2 + \cdots + r_{k_a}}{k_a}$$

"sample average"

$$\lim_{k_a \to \infty} Q_t(a) = Q^*(a)$$

What is the behaviour with finite samples?

ε-Greedy Action Selection

- Greedy action selection:

$$a_t = a_t^* = \arg\max_a Q_t(a)$$

- ε-Greedy:

$$a_t = \begin{cases} a_t^* & \text{with probability } 1-\varepsilon \\ \text{random action} & \text{with probability } \varepsilon \end{cases}$$

. . . the simplest way to balance exploration and exploitation

A simple bandit algorithm

Initialize, for $a = 1$ to k:

$Q(a) \leftarrow 0$

$N(a) \leftarrow 0$

Repeat forever:

$$A \leftarrow \begin{cases} \arg\max_a Q(a) & \text{with probability } 1-\varepsilon \quad \text{(breaking ties randomly)} \\ \text{a random action} & \text{with probability } \varepsilon \end{cases}$$

$R \leftarrow bandit(A)$

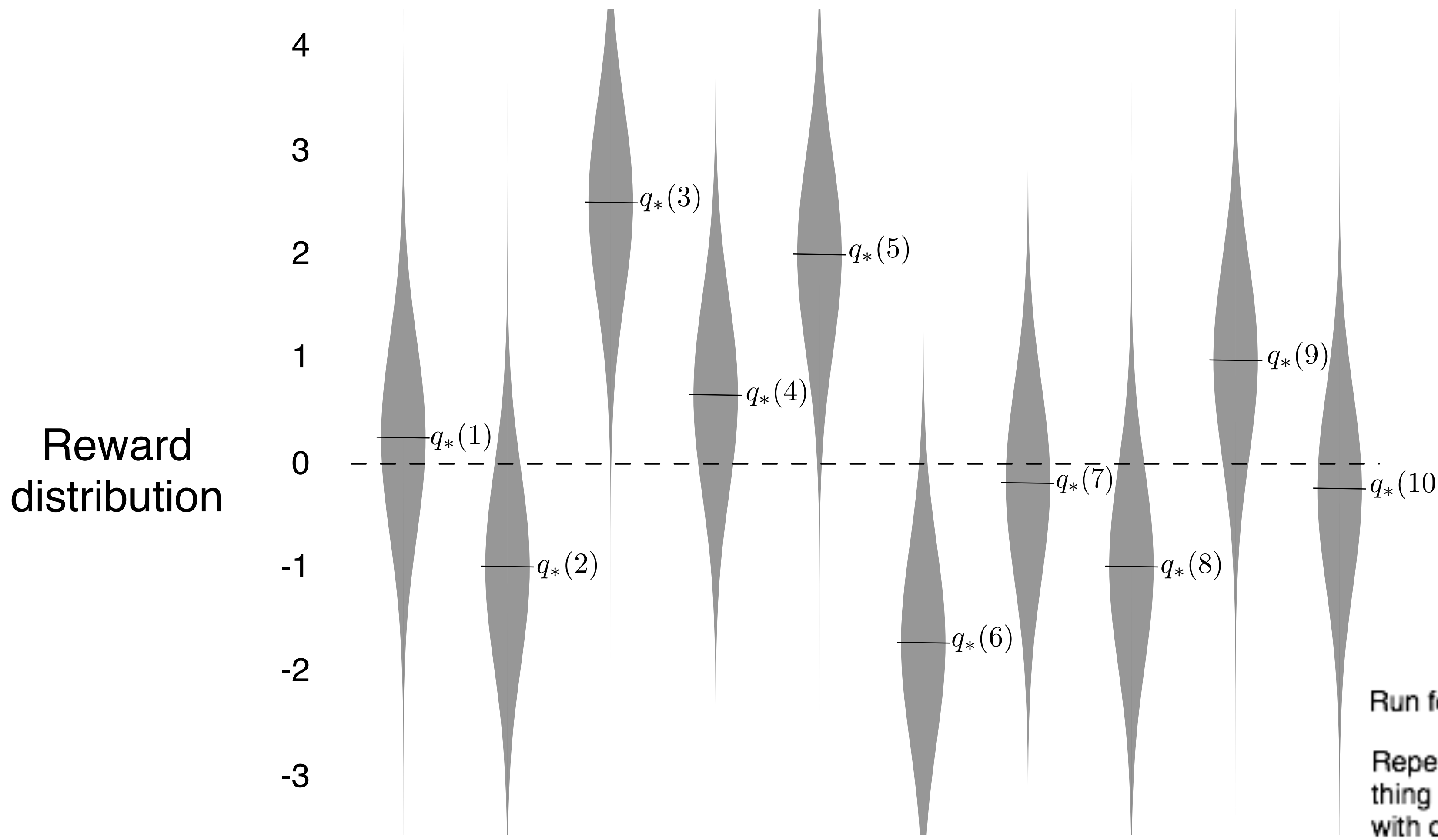
$N(A) \leftarrow N(A) + 1$

$Q(A) \leftarrow Q(A) + \frac{1}{N(A)}\big[R - Q(A)\big]$

Worked Example: 10-Armed Testbed

- n = 10 possible actions
- Each $Q^*(a)$ is chosen randomly from a normal distrib.: $N(0,1)$
- Each r_t is also normal: $N(Q^*(a_t),1)$
- 1000 plays, repeat the whole thing 2000 times and average the results

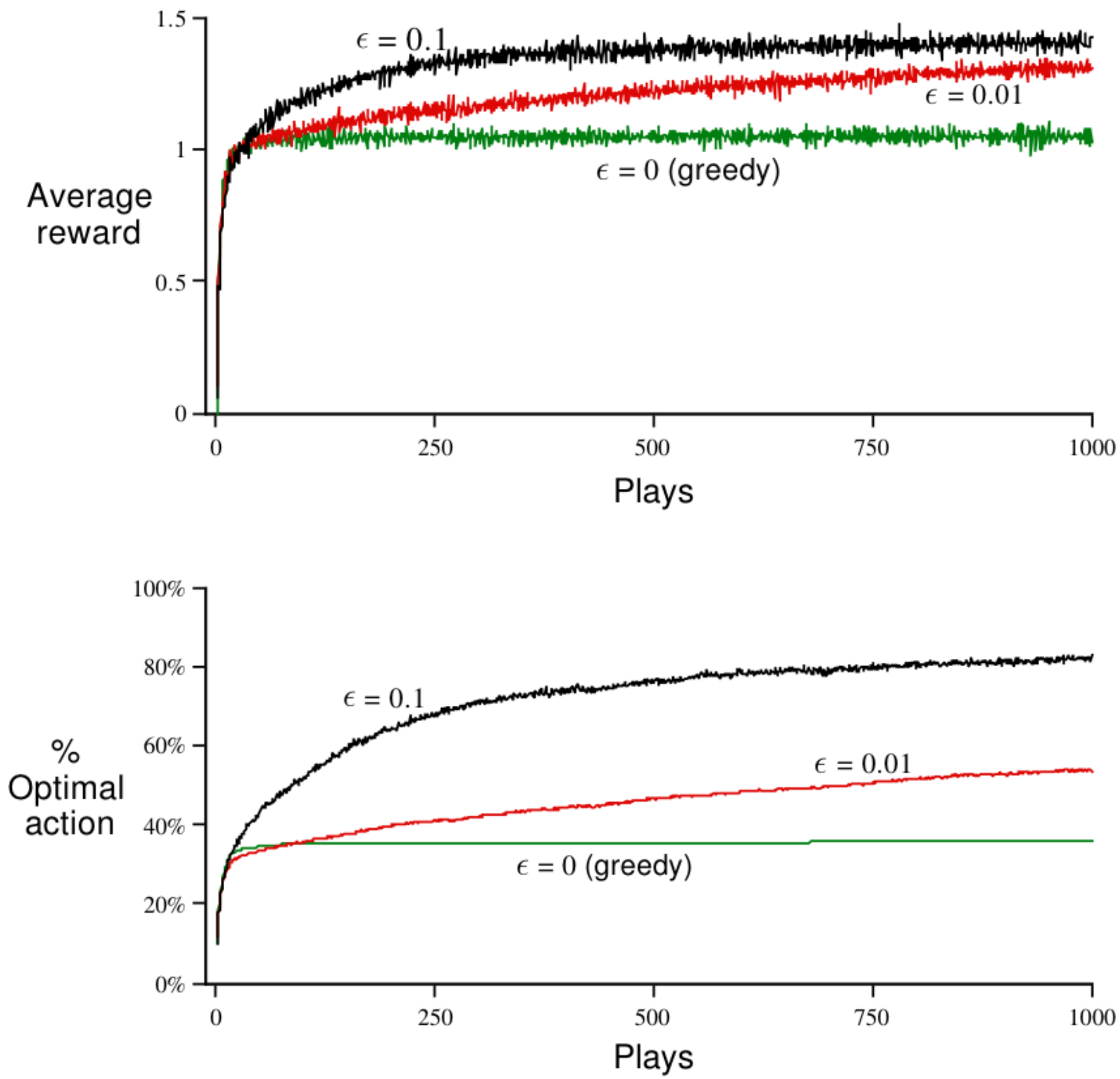
10-Armed Testbed Rewards



Run for 1000 steps

Repeat the whole thing 2000 times with different bandit tasks

ε-Greedy Methods on the 10-Armed Testbed



Reasoning further: Interval Estimation

- Attribute to each arm an "optimistic initial estimate" within a certain confidence interval
- Greedily choose arm with highest optimistic mean (upper bound on confidence interval)

- Infrequently observed arm will have over-valued reward mean, leading to exploration
- Frequent usage pushes optimistic estimate to true values

Interval Estimation (IE) Procedure

- Associate to each arm 100(1-α)% reward mean upper band

- Assume, e.g., rewards are normally distributed
- Arm is observed n times to yield empirical mean & std. dev.
- α-upper bound (u_α) can be written in terms of mean μ and standard deviation σ as:

$$u_\alpha = \hat{\mu} + \frac{\hat{\sigma}}{\sqrt{n}} c^{-1}(1-\alpha)$$

$$c(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{x^2}{2}\right) dx$$

c is the Cum. Distribution Function

Visualize on the board …

How to Evaluate an Online Alg.: Regret

- After you have played for T rounds, you experience a regret:
  = [Reward sum of optimal strategy] – [Sum of actual collected rewards]

$$\rho = T\mu^* - \sum_{t=1}^{T} \hat{r}_t = T\mu^* - \sum_{t=1}^{T} E\left[r_{i_t}(t)\right]$$

$$\mu^* = \max_k \mu_k$$

Randomness in draw of rewards & Player's strategy

- If the average regret per round goes to zero with probability 1, asymptotically, we say the strategy has **no-regret** property

  ~ guaranteed to converge to an optimal strategy
- ε-greedy is sub-optimal (so has some regret). Why?
- If α is carefully controlled, IE could be zero-regret strategy

Moving Back to the MDP Model

- We have actions (a_t) as well as states (s_t)
- System dynamics are stochastic – represented by a probability distribution for transitions between states
- Problem is defined as maximization of expected rewards
  - Recall that $E(X) = \Sigma\, x_i p(x_i)$ for finite-state systems

State Transition Dynamics:

$$\mathcal{P}^a_{ss'} = Pr\{s_{t+1} = s' | s_t = s, a_t = a\}$$

Expected Rewards:

$$\mathcal{R}^a_{ss'} = E\{r_{t+1} | s_t = s, a_t = a, s_{t+1} = s'\}$$

Note that:

$$\mathcal{R}^a_s = \sum_{s'} \mathcal{P}^a_{ss'} \mathcal{R}^a_{ss'}$$

Decision Criterion

What is the criterion for optimization (i.e., learning)?

$$R_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1},$$

where $\gamma, 0 \le \gamma \le 1,$ is the **discount rate**.

What would be the effect of changing γ?

Expectation of this Criterion: Value Functions

- Value functions are used to determine how good it is for the agent to be in a given state (sometimes also to perform an action at s)
  - Expectation of the criterion in prev. slide (similar to "cost-to-go")
- This is defined w.r.t. a specific policy, i.e., action distribution $\pi(s,a)$

State value function:

$$V^{\pi}(s) = E_{\pi}\{R_t | s_t = s\} = E_{\pi}\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | s_t = s\}$$

Value Functions

Note that there are multiple sources of (probabilistic) uncertainty:

- In state s, one is allowed to select different actions a
- The system may transition to different states s' from s
- Depending on the above, return (defined in terms of reward) is a random variable – which we seek to maximize in expectation

$$
\begin{aligned}
V^{\pi}(s) &= E_{\pi}\{R_t | s_t = s\} \\
&= E_{\pi}\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | s_t = s\} \\
&= E_{\pi}\{r_{t+1} + \gamma \sum_{k=0}^{\infty} \gamma^k r_{t+k+2} | s_t = s\}
\end{aligned}
$$

Recursive Form of V – **Bellman Equation**

$V^{\pi}(s) = E_{\pi}\{r_{t+1} + \gamma \sum_{k=0}^{\infty} \gamma^k r_{t+k+2} | s_t = s\}$

We rewrite as follows:

Expand 1-step forward & rewrite expectation

- First term: $\sum_a \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^a \mathcal{R}_{ss'}^a$
- Second term: $\sum_a \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^a \gamma E_{\pi}\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+2} | s_{t+1} = s'\}$

$\therefore V^{\pi}(s) = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma E_{\pi}\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+2} | s_{t+1} = s'\}]$

$V^{\pi}(s) = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V^{\pi}(s')]$

Recursive Expression for an *Optimal* Value Function, V^*

- A key result for Dynamic Programming with Markov Decision Process Models is the following recursive expression that holds true for each state:

$$V^*(s) = \max_a \sum_{s'} \mathcal{P}^a_{ss'} [\mathcal{R}^a_{ss'} + \gamma V^*(s')]$$

- This is a characterization of the value function, i.e., when you find an optimal value function then each state and its neighbours will satisfy this recursive relationship

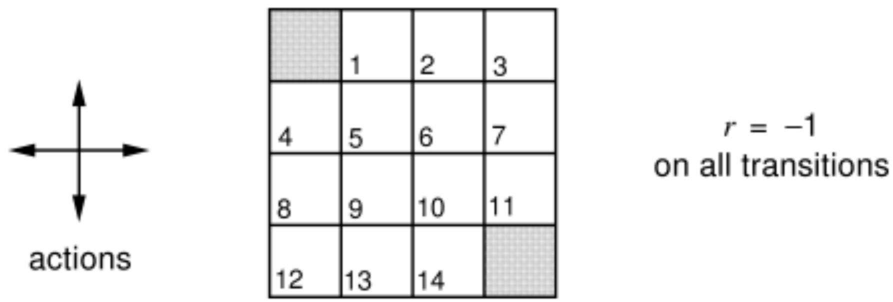
Given a Policy, what is Value Function, V?

Solve iteratively, with a sequence of value functions, $V_0, V_1, V_2, \ldots : \mathcal{S} \mapsto \Re$

$$V_{k+1}(s) = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V_k(s')\right] \; \forall s \in \mathcal{S}$$

$V_k = V^\pi$ is a fixed-point for these updates, as $k \to \infty$
- *Iterative* policy evaluation.

Grid-World Example



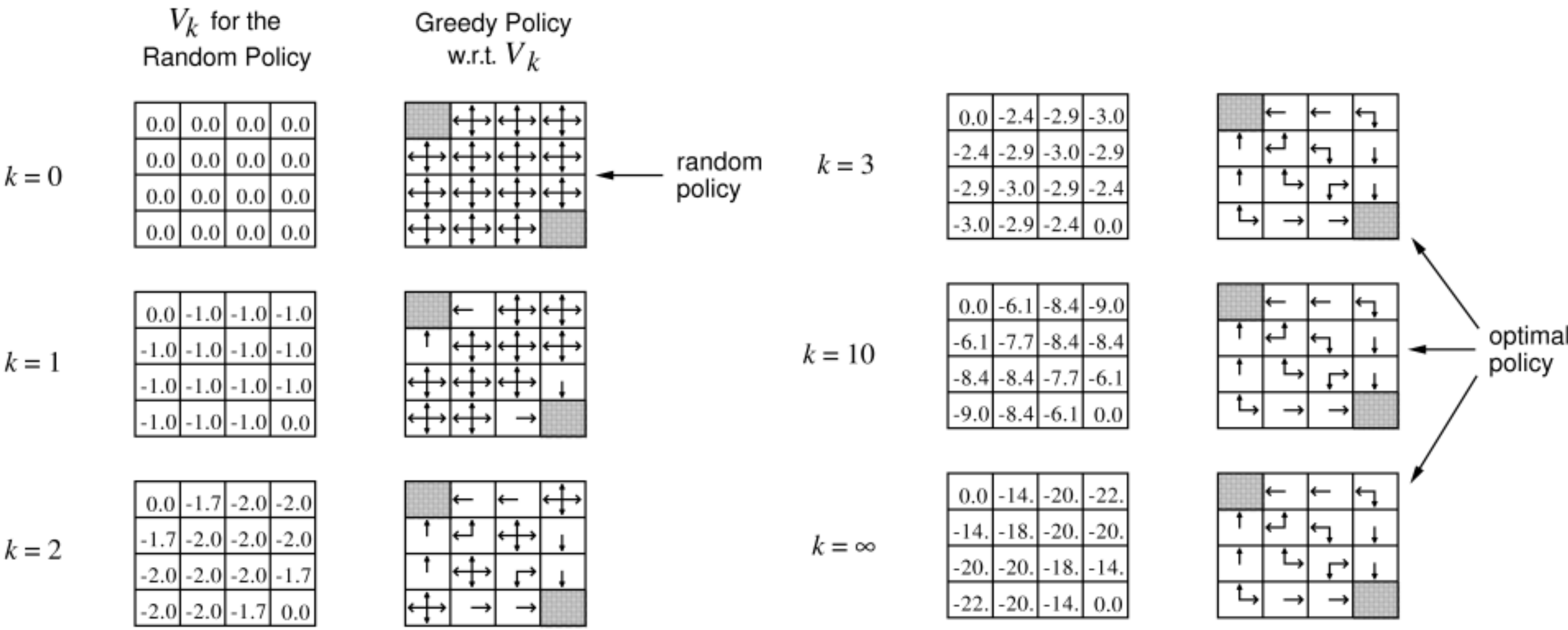
Four possible actions: $\mathcal{A} = \{$ up, down, right, left$\}$

- the actions change state deterministically (but, not allowed to go off grid)

Encoded in transition probabilities, e.g., $\mathcal{P}_{5,6}^{\text{right}} = 1$, $\mathcal{P}_{5,10}^{\text{right}} = 0$, $\mathcal{P}_{7,7}^{\text{right}} = 1$

Undiscounted, episodic task with reward -1 everywhere except goal states.

Iterative Policy Evaluation in Grid World



Note: *The value function can be searched greedily to find long-term optimal actions*

Now, Given a Value Function, Can We Improve a Policy?

- Yes, compute the following:

$$\pi'(s) = \arg\max_a E\{r_{t+1} + \gamma V^{\pi}(s_{t+1}|s_t = s, a_t = a\}$$

$$\pi'(s) = \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'}[\mathcal{R}^a_{ss'} + \gamma V^{\pi}(s')]$$

- ... and this can be iterated upon!

$$\pi_0 \xrightarrow{E} V^{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} V^{\pi_1} \xrightarrow{I} \pi_2 \xrightarrow{E} V^{\pi_2} \ldots \xrightarrow{I} \pi^* \xrightarrow{E} V^*$$

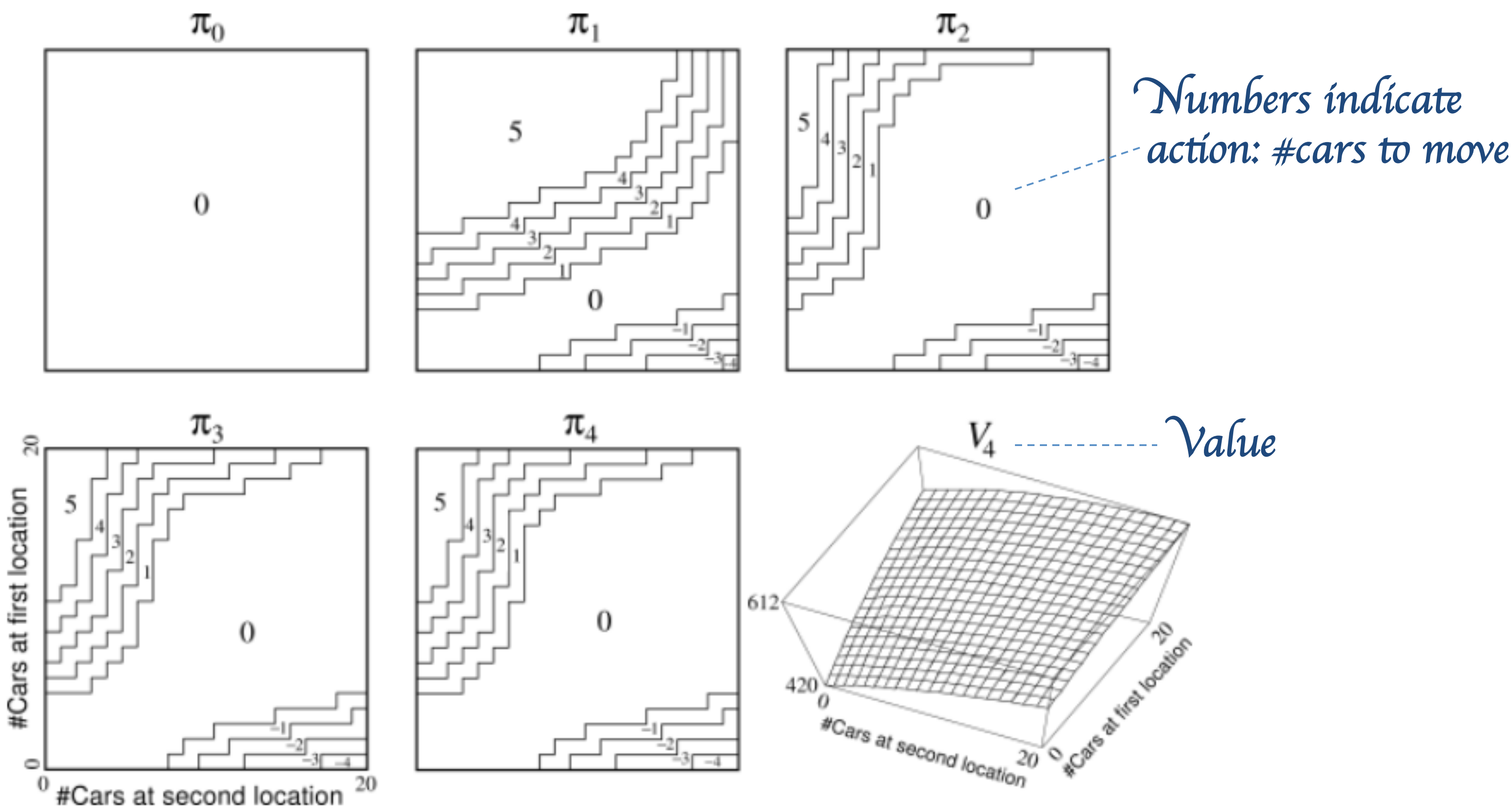
Example: Jack's Car Rental

- £10 for each car rented (must be available when request received)
- Two locations, maximum of 20 cars at each
- Cars returned and requested randomly
  - Poisson distribution, n returns/requests with probability $\frac{\lambda^n}{n!}e^{-\lambda}$
  - Location 1: Average requests = 3, Average returns = 3
  - Location 2: Average requests = 4, Average Returns = 2
- Can move up to 5 cars between locations overnight (costs £2 each)

Problem setup:

- States, actions, rewards?
- Transition probabilities?

Solution: Jack's Car Rental



Points to Ponder: Jack's Car Rental

- Suppose first car moved is free but all others transfers cost £2
  - From Location 1 to Location 2 (not other direction!)
  - Because an employee would anyway go in that direction, by bus

- Suppose only 10 cars can be parked for free at each location
  - More than 10 incur fixed cost £4 for using an extra parking lot

… typical examples of '*real-world nonlinearities*'

Value Iteration

Each step in Policy Iteration needs Policy Evaluation (upto convergence)
- can we avoid this computational overhead?

Just update the values for *one* iteration and then improve the policy.

Update rule:

$$V = \max_a \sum_{s'} P^a_{ss'} [R^a_{ss'} + \gamma V(s')]$$

… use Bellman equation as update rule

So we sweep through the state space once (and don't wait for V to stop changing, as in policy evaluation), then improve the policy, then repeat.

This update combines the one-iteration update of V plus the policy improvement (greedification wrt V) in one step.

Acknowledgements

The main source for this section is Sutton+Barto, Reinforcement Learning:

- Part 1 [MAB]: Ch 2 (sections 2.1-2.2)
- Part 2 [Bellman/Value]: Ch 3,4 (sections 3.7-3.8, 4.1-4.4)

The interval estimation procedure is from L. Pack Kaelbling, *Learning in Embedded Systems*, MIT Press (Ch 4)

https://books.google.co.uk/books?id=WiN53ZYd0kgC&dq=leslie+kaelbling+interval+estimation&source=gbs_navlinks_s