Decision Making

in Robots and Autonomous Agents

Control: How should a robot stay "in place"?

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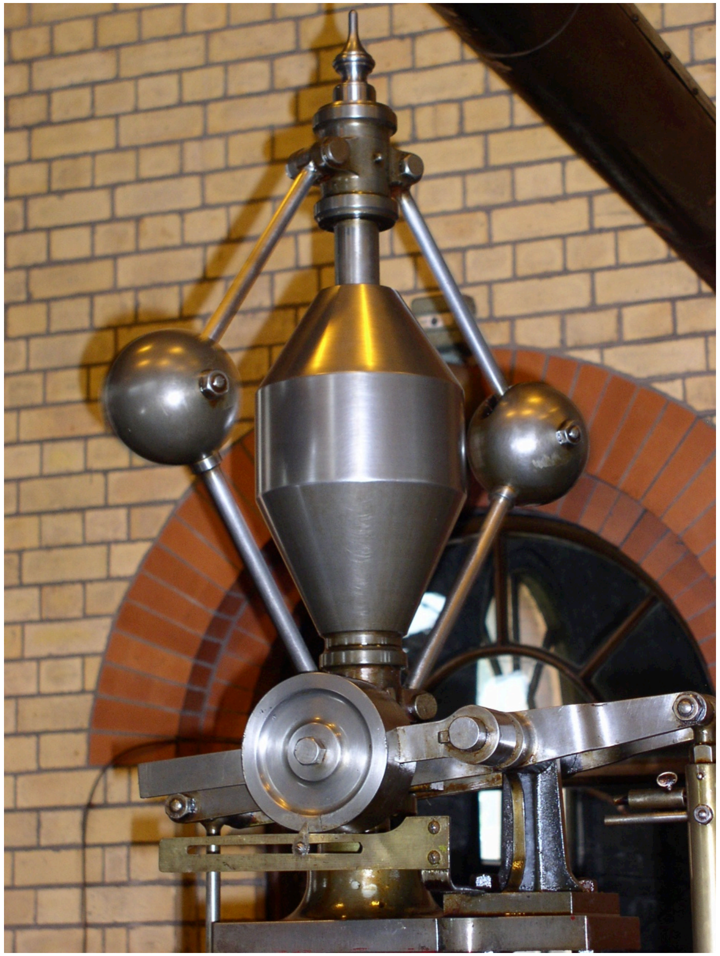
Objectives of this Lecture

- Give a selective recap of key ideas from control theory, as a very first approach to the "synthesis of robot motion"
  - If you have studied control before, you should recognize the concepts although the narrative may still be new
  - If you have not studied control before, this should give you useful background that will help contextualize other concepts to come later

- After a first half surveying a few key concepts, we will spend the second half of the lecture thinking concretely about the design of a controller for one particular model system: inverted pendulum

Centrifugal Governor (James Watt, 1788)



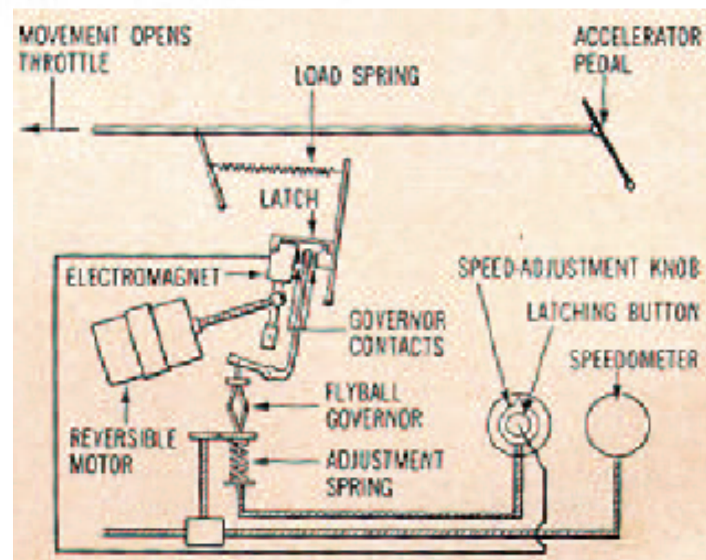
Not Only of Historical Interest…

. . . with AUTO PILOT for *CHRYSLER and IMPERIAL*

Just set the convenient instrument panel dial to your desired speed. Then drive in your usual manner. When you reach the pre-set speed you feel a gentle nudge of the accelerator on your foot telling you you've reached your desired speed.

For completely automatic control, pull the control knob when you feel the nudge of the pedal and remove your foot from the accelerator. Then, drive relaxed with your eyes on the road.

A touch of your brake pedal instantly returns the control to manual. To return to automatic control, just accelerate until you feel the nudge and remove your foot from the accelerator.



How does a Governor Work? Proportional Control

- A feedback system that controls the speed of an engine by regulating the amount of fuel (or working fluid) admitted
- Goal is to maintain a near-constant speed, irrespective of the load or fuel-supply conditions.

A sequence of operations:

1) Power is supplied to the governor from the engine's output shaft. The governor is connected to a throttle valve that regulates the flow of working fluid (steam) supplying the prime mover.

How does a Governor Work? Proportional Control

2) As the speed of the prime mover increases, the central spindle of the governor rotates at a faster rate and the kinetic energy of the balls increases.

3) This allows the two masses on lever arms to move outwards and upwards against gravity.

4) If the motion goes far enough, this motion causes the lever arms to pull down on a thrust bearing, which moves a beam linkage, which reduces the aperture of a throttle valve.

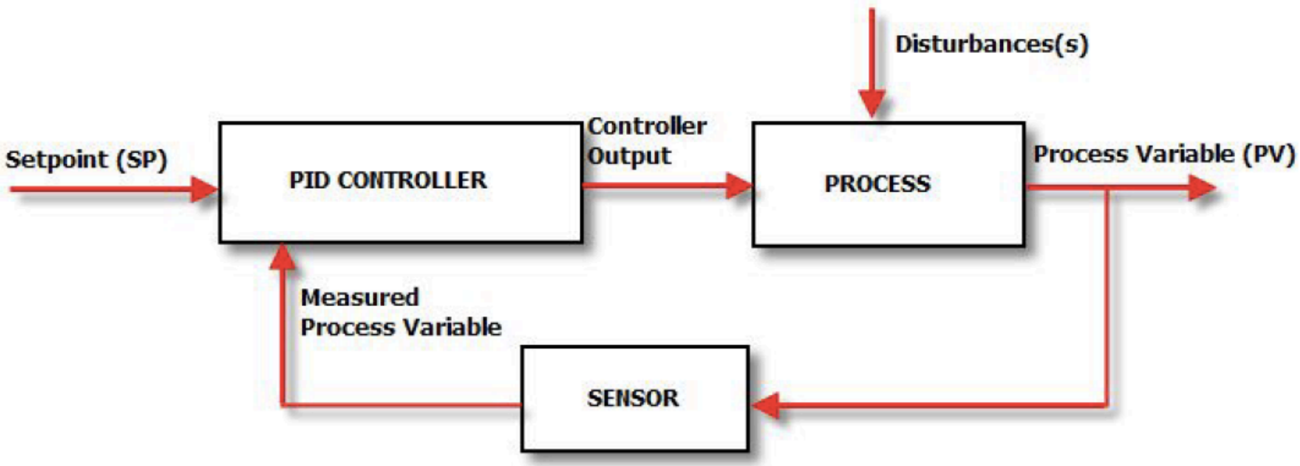
5) The rate of working-fluid entering the cylinder is thus reduced and the speed of the prime mover is controlled, preventing over-speeding.

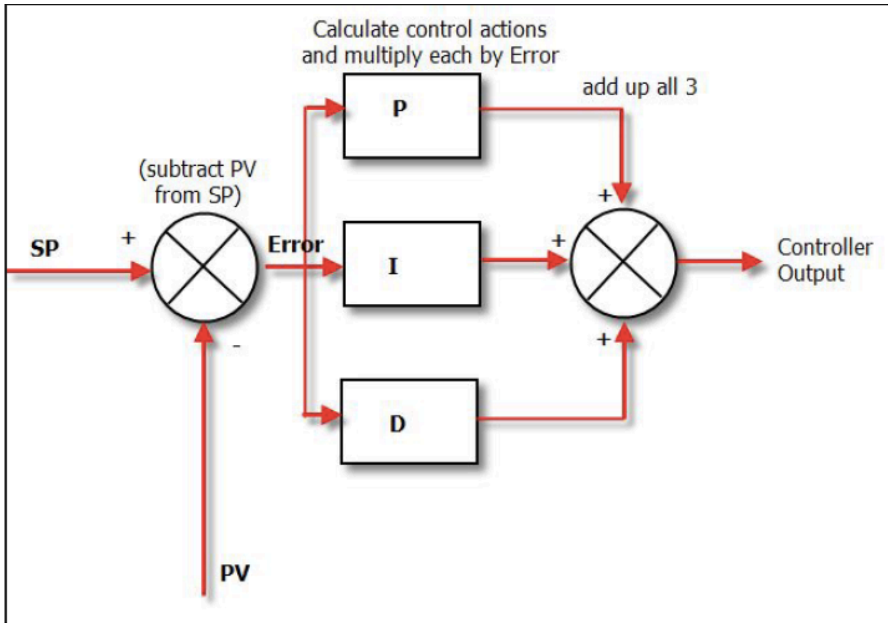
Proportional Control

- We want to hold system "in place" – in this case, at a certain rate of flow
- When flow exceeds desired value, the mechanism applies a correction which is proportional to the excess

- This idea of regulation is quite valuable in all engineered systems
- However, the quantity being regulated is not always flow
- How to write down the principle mathematically?
  - We also need to say how to describe the system

PID Controllers





Proportional-Integral-Derivative Control

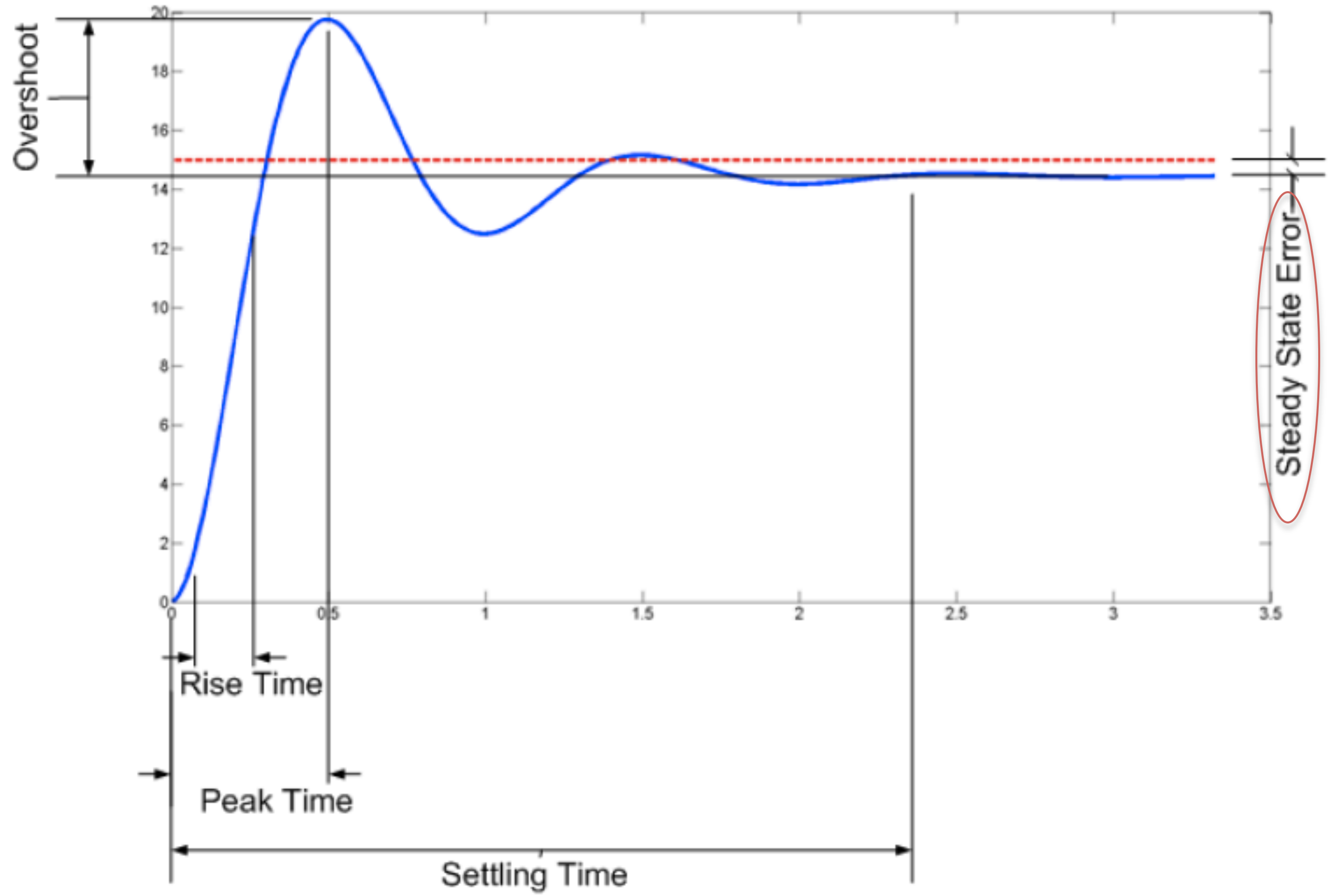
- The control signal, $u(t)$, is given in terms of the error $e(t)$ as,

$$u(t) = K_p e(t) + K_i \int_{t_0}^{t} e(\tau) d\tau + K_d \dot{e}(t)$$

- This simple algorithm is most useful when processes are known to be stable and not very oscillatory
  - Parameters may not be well known, however

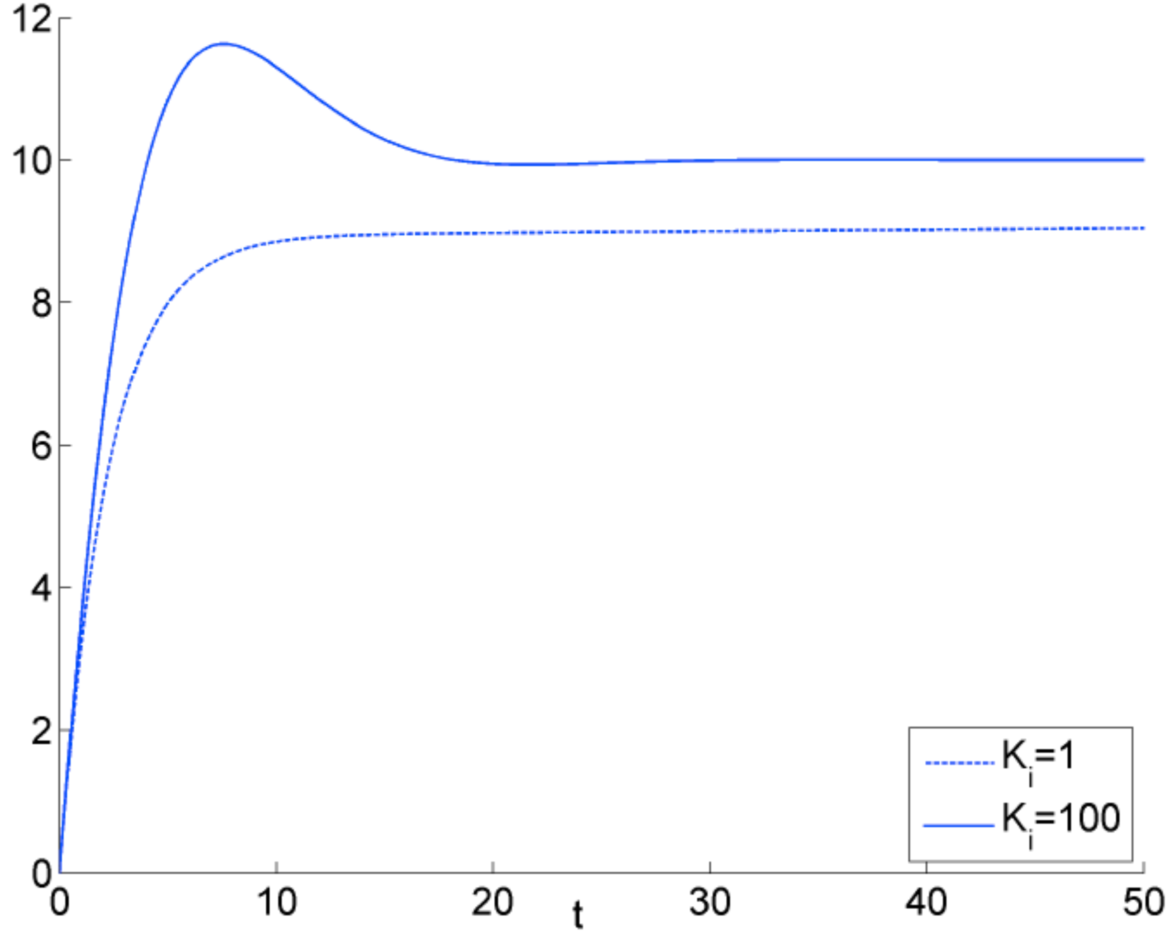
- Why is each term needed?
- How could we set the scale factors (the Ks)?

Typical Step Response of 2nd Order System with Proportional Control



Why?

Step Response with Different Levels of Integral Gain (Setpoint = 10)



Effects of Different Components

Control Action	Rise Time	Overshoot	Settling Time	Steady State Error
Increasing K_p	reduces	increases	small change	reduces
Increasing K_i	reduces	increases	increases	eliminates
Increasing K_d	small change	reduces	reduces	small change

Many Design Heuristics, e.g., Ziegler-Nichols Rules (1942)

- Trial and error procedure, entirely empirical
- Gradually increase proportional gain alone until the system begins to oscillate (with loop gain, K_u, and period, T_u)
- Then, set the gains to be:

$$K_p = \frac{1}{2} K_u$$

$$K_i = \frac{2}{T_u} K_p$$

$$K_d = \frac{T_u}{8} K_p$$

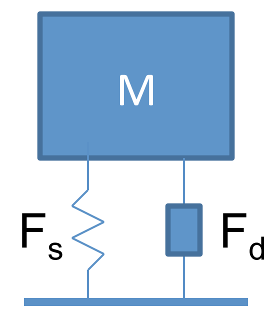
- How to think about design and dynamics, more generally?

Linear Time Invariant (LTI) Systems

- Consider the simple spring-mass-damper system:
- The force applied by the spring is $F_s = -kz(t)$
- Correspondingly, for the damper: $F_d = \gamma\dot{z}(t)$
- The combined equation of motion of the mass becomes:

$$x\ddot{z}(t) = -\gamma\dot{z}(t) - kz(t)$$

- One could also express this in state space form:



$$x(t) = [x_1(t), x_2(t)]' = [z(t), \dot{z}(t)]'$$

$$\dot{x}(t) = \begin{pmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -\frac{1}{m}(\gamma x_2(t)) + kx_1(t) \end{pmatrix}$$

Linear ODE

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \frac{\gamma}{m} \end{pmatrix} x(t) = Ax(t)$$

Solution of a Linear ODE

$\dot{x} = kx$, $x \in \mathbb{R}$

For initial condition $\phi(0) = x_0$, the solution is $\phi(t) = e^{kt}x_0$

i.e., time evolution of state is given by operator $g^t = e^{kt}$, with velocity $\mathbf{v} = kt$

This type of "exponential term" is a feature of all linear dynamical systems

The multivariate case $x(t) = e^{A(t-t_0)}x_0$

$$\begin{aligned} \boxed{e^{A(t-t_0)}} &= \sum_{i=0}^{\infty} \frac{A^i(t-t_0)^i}{i!} \\ &= I_{n\times n} + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \ldots \end{aligned}$$

This is state transition matrix $\phi(t)$:
In linear algebra, there are numerous ways to compute this…

Example

Determine the matrix exponential, and hence the state transition matrix, and the homogeneous response to the initial conditions $x_1(0) = 2$, $x_2(0) = 3$ of the system with state equations:

$$\begin{aligned} \dot{x}_1 &= -2x_1 + u \\ \dot{x}_2 &= x_1 - x_2. \end{aligned}$$

The system matrix is

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}.$$

Example, contd.

$$\begin{aligned}
\mathbf{\Phi}(t) &= e^{\mathbf{A}t} \\
&= \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \ldots + \frac{\mathbf{A}^kt^k}{k!} + \ldots\right) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} t + \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \frac{t^2}{2!} \\
&\qquad + \begin{bmatrix} -8 & 0 \\ 7 & -1 \end{bmatrix} \frac{t^3}{3!} + \ldots \\
&= \begin{bmatrix} 1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!} + \ldots & 0 \\ 0 + t - \frac{3t^2}{2!} + \frac{7t^3}{3!} + \ldots & 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots \end{bmatrix}.
\end{aligned}$$

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

Example, contd.

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$

$$\begin{aligned} x_1(t) &= x_1(0)e^{-2t} \\ x_2(t) &= x_1(0)\left(e^{-t} - e^{-2t}\right) + x_2(0)e^{-t}. \end{aligned}$$

$$\begin{aligned} x_1(t) &= 2e^{-2t} \\ x_2(t) &= 2\left(e^{-t} - e^{-2t}\right) + 3e^{-t} \\ &= 5e^{-t} - 2e^{-2t}. \end{aligned}$$

Basic Notion: Stability

- Simple question:

  Given the system, $\dot{x}(t) = Ax(t)$

  where in phase space, $(x, \dot{x})$, will it come to rest?

Any guesses?
Think about solution in previous slide…

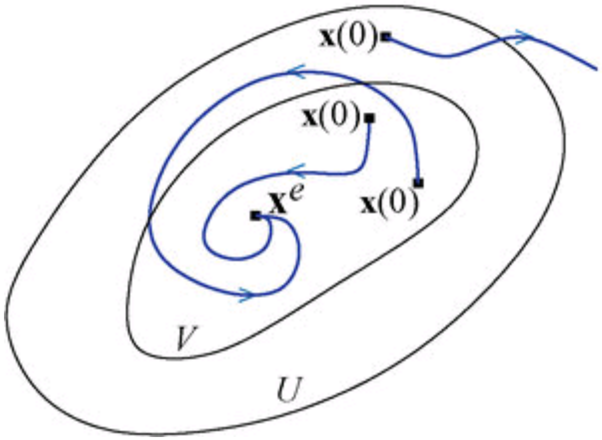
Do you know what this is? (Whiteboard)

- This point is called the equilibrium point
  - If initialized there, dynamics will not take it away
  - If perturbed, system will eventually return and stay there

Stability

An equilibrium position $x = 0$ is *stable* (in Lyapunov's sense) if given $\epsilon > 0$, $\exists\delta > 0$ (not dependent on t), s.t. $\forall x_0, |x_0| < \delta$ the solution satisfies $|\phi(t)| < \epsilon$, $\forall t > 0$

Asymptotic stability: Lyapunov stabile and $\lim_{t\to+\infty}\phi(t) = 0$



Stability for an LTI System, $\dot{x}(t) = Ax(t)$

Unforced (homogeneous) response: $x_i(t) = \sum_{j=1}^{n} m_{ij} e^{\lambda_j t}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

$$\mathbf{x_h}(t) = \mathrm{M} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

Stability for an LTI System

If you differentiate the homogeneous response, $\frac{dx_i}{dt} = \sum_{j=1}^{n} \lambda_j m_{ij} e^{\lambda_j t}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 m_{11} & \lambda_2 m_{12} & \dots & \lambda_n m_{1n} \\ \lambda_1 m_{21} & \lambda_2 m_{22} & \dots & \lambda_2 m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 m_{n1} & \lambda_2 m_{n2} & \dots & \lambda_n m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}.$$

The system being considered is $\dot{x}(t) = Ax(t)$, so:

$$\begin{bmatrix} \lambda_1 m_{11} & \lambda_2 m_{12} & \dots & \lambda_n m_{1n} \\ \lambda_1 m_{21} & \lambda_2 m_{22} & \dots & \lambda_2 m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 m_{n1} & \lambda_2 m_{n2} & \dots & \lambda_n m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

LTI Stability, in algebraic equations

- The above equation leads to an eigenvalue problem:

$$\lambda_i \mathbf{m}_i = \mathbf{A}\mathbf{m}_i \qquad\qquad i = 1, 2, \ldots, n.$$

$$[\lambda_i \mathbf{I} - \mathbf{A}]\,\mathbf{m}_i = 0$$

- For this to have nontrivial solutions:

Characteristic eqn.

$$\Delta(\lambda_i) = \det[\lambda_i \mathbf{I} - \mathbf{A}] = 0.$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_1\lambda + a_0 = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = 0.$$

Stability: LTI System, $\dot{x}(t) = Ax(t)$

Theorem. *Let* λ_i, $i \in \{1, 2, \ldots, n\}$ *denote the eigenvalues of A. Let* $re(\lambda_i)$ *denote the real part of* λ_i*. Then the following holds:*

1. $x_e = 0$ *is stable if and only if* $re(\lambda_i) \leq 0$, $\forall i$
2. $x_e = 0$ *is asymptotically stable if and only if* $re(\lambda_i) < 0$, $\forall i$
3. $x_e = 0$ *is unstable if and only if* $re(\lambda_i) > 0$, *for some* i

For the spring-mass-damper example, the eigenvalues are:

$$\frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

With positive damping, we get asymptotic stability

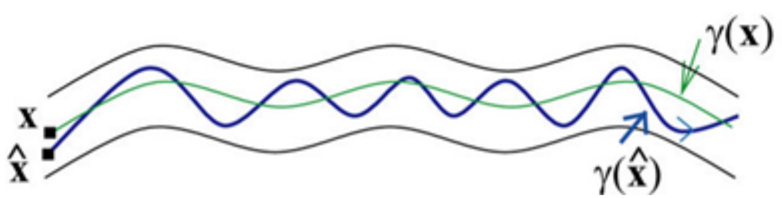
Exercise (ponder at home)

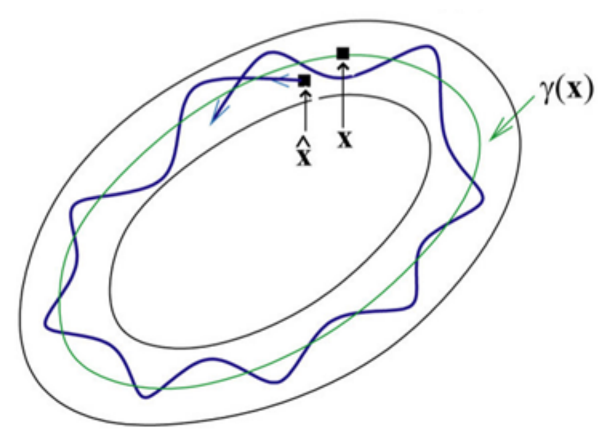
Can you visualize (i.e., draw the curve vs. time) state variables for the case of asymptotic stability, instability and the borderline in between?

Other Related Notions: Orbital Stability

- Stability doesn't only refer to being at rest at a point
  - could be defined in terms of staying in a subset, e.g., path

Definition. *An orbit $\gamma(x)$ is orbitally stable if for any $\epsilon > 0$, there is a neighbourhood V of x so that for all $\hat{x}$ in V, γx and $\gamma\hat{x}$ are ϵ-close.*
Loosely speaking, $|\gamma(x) - \gamma(\hat{x})| < \epsilon$ at all times.





Is Stability Really an Issue?

Some Aircrafts are *Designed* to be Statically Unstable! Why?

Jump to 2:07 for exciting bit!

[https://www.youtube.com/watch?v=2CUyoi634wc]

Some Acknowledgements

- Slide 3: By Globbet - Own work by uploader, with permission from the Mill Meece Pumping Station Preservation Trust, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=7421535

- Slide 4: http://www.cds.caltech.edu/~murray/cdspanel/