Discrete Mathematics & Mathematical Reasoning
Cardinality

Colin Stirling

Informatics

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$f : Even \rightarrow \mathbb{N}$ with $f(2n) = n$ is a bijection

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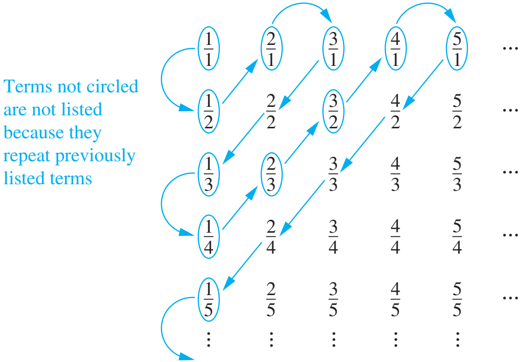

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- First define an (alphabetical) ordering on the symbols in Σ Show that the strings can be listed in a sequence
  - First single string ε of length 0
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The set of Java-programs is countable; a program is just a finite string

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- With the property $d_m = d(m)$ is the mth symbol

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Let X be the set of infinite binary strings. For a contradiction assume that a bijection $f : \mathbb{Z}^+ \to X$ exists. So, f must be onto (surjective). Assume that $f(i) = d^i$ for $i \in \mathbb{Z}^+$. So, $X = \{d^1, d^2, \ldots, d^m, \ldots\}$. Define the infinite binary string d as follows: $d_n = (d^n_n + 1) \bmod 2$. But for each m, $d \neq d^m$ because $d_m \neq d^m_m$. So, f is not a surjection. $\square$

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Similar argument shows that $\mathbb{R}$ via $[0, 1]$ is uncountable using infinite decimal strings (see book)

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The set of functions $F = \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$ *is uncountable*

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Therefore, “most functions” in F are not computable!

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- Example $|(0,1)| = |(0,1]|$
- $|(0,1)| \leq |(0,1]|$ using identity function
- $|(0,1]| \leq |(0,1)|$ use $f(x) = x/2$ as $(0,1/2] \subset (0,1)$

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Proof.

Consider the injection $f : A \to \mathcal{P}(A)$ with $f(a) = \{a\}$ for any $a \in A$. Therefore, $|A| \leq |\mathcal{P}(A)|$. Next we show there is not a surjection $f : A \to \mathcal{P}(A)$. For a contradiction, assume that a surjection f exists. We define the set $B \subseteq A$: $B = \{x \in A \mid x \notin f(x)\}$. Since f is a surjection, there must exist an $a \in A$ s.t. $B = f(a)$. Now there are two cases:

1. If $a \in B$ then, by definition of B, $a \notin B = f(a)$. Contradiction
2. If $a \notin B$ then $a \notin f(a)$; by definition of B, $a \in B$. Contradiction

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