# Discrete Mathematics & Mathematical Reasoning
# Predicates, Quantifiers and Proof Techniques

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- Disjunction: $\vee$
- Implication: $\rightarrow$
- Biconditional: $\leftrightarrow$

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The meaning of logical connectives can be defined using truth tables

# Examples of logical implication and equivalence

- $(p \wedge (p \rightarrow q)) \rightarrow q$
- $(p \wedge \neg p) \rightarrow q$
- $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
- $\vdots$

# Examples of logical implication and equivalence

- $(p \land (p \to q)) \to q$
- $(p \land \neg p) \to q$
- $((p \to q) \land (q \to r)) \to (p \to r)$
- $\vdots$
- $(p \to q) \leftrightarrow (\neg q \to \neg p)$
  Contraposition
- $\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$ De Morgan
- $\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)$ De Morgan
- $\neg(p \to q) \leftrightarrow (p \land \neg q)$
- $\vdots$

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We need a language to talk about objects, their properties and their relations

# Sansa the cat (with whiskers)

# Formally same argument as

Given the following two premises

- All students in this class understand logic
- Colin is a student in this class

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it follows that

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# Predicate logic

Extends propositional logic by the new features

- Variables: $x$, $y$ ,$z$, $\ldots$
- Predicates: $P(x)$, $Q(x)$, $R(x, y)$, $M(x, y, z)$, $\ldots$
- Quantifiers: $\forall$, $\exists$

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Predicates are a generalisation of propositions

- Can contain variables $M(x, y, z)$
- Variables stand for (and can be replaced by) elements from their domain
- The truth value of a predicate depends on the values of its variables

# Examples

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- $P(8)$ is true
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$S(x_1, \ldots, x_{11}, y)$ is "$x_1 + \ldots + x_{11} = y$"

- $S(1, 2, \ldots, 11, 66)$ is true

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- A formula that does not contain any free variables is a proposition and has a truth value

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- We always assume that a domain is nonempty

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  So, ($C(Sansa) \rightarrow W(Sansa)$ and $C(Sansa)$) implies $W(Sansa)$

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- $n^2$ has the form for some $m$, $n^2 = 2m + 1$; so $Q(n)$

# Any odd integer is the difference of two squares

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- That is, assume $\neg B(c)$ then show $\neg A(c)$
- Use the definition/properties of $\neg B(c)$

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Proof Let $n, m \in \mathbb{Z}$ be arbitrary. We will prove that if $n$ and $m$ do not have the same parity then $n + m$ is odd. Without loss of generality we assume that $n$ is odd and $m$ is even, that is $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $m = 2\ell$ for some $\ell \in \mathbb{Z}$. But then $n + m = 2k + 1 + 2\ell = 2(k + \ell) + 1$. And thus $n + m$ is odd. Now by equivalence of a statement with it contrapositive derive that if $n + m$ is even, then $n$ and $m$ have the same parity.

If $n = ab$ where $a, b$ are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

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- Therefore, $\neg\neg p$ which is equivalent to $p$

# $\sqrt{2}$ is irrational

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Proof Assume towards a contradiction that $\sqrt{2}$ is rational, that is there are integers $a$ and $b$ with no common factor other than 1, such that $\sqrt{2} = a/b$. In that case $2 = a^2/b^2$. Multiplying both sides by $b^2$, we have $a^2 = 2b^2$. Since $b$ is an integer, so is $b^2$, and thus $a^2$ is even. As we saw previously this implies that $a$ is even, that is there is an integer $c$ such that $a = 2c$. Hence $2b^2 = 4c^2$, hence $b^2 = 2c^2$. Now, since $c$ is an integer, so is $c^2$, and thus $b^2$ is even. Again, we can conclude that $b$ is even. Thus $a$ and $b$ have a common factor 2, contradicting the assertion that $a$ and $b$ have no common factor other than 1. This shows that the original assumption that $\sqrt{2}$ is rational is false, and that $\sqrt{2}$ must be irrational.

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Proof Suppose towards a contradiction that there are only finitely many primes $p_1, p_2, p_3, \ldots, p_k$. Consider the number $q = p_1p_2p_3 \ldots p_k + 1$, the product of all the primes plus one. By hypothesis $q$ cannot be prime because it is strictly larger than all the primes. Thus, by the lemma, it has a prime divisor, $p$. Because $p_1, p_2, p_3, \ldots, p_k$ are all the primes, $p$ must be equal to one of them, so p is a divisor of their product. So we have that $p$ divides $p_1p_2p_3 \ldots p_k$ , and $p$ divides $q$, but that means $p$ divides their difference, which is 1. Therefore $p \leq 1$. Contradiction. Therefore there are infinitely many primes.

# Proof by cases

- To prove a conditional statement of the form

$$(p_1 \vee \cdots \vee p_k) \rightarrow q$$

- Use the tautology

$$((p_1 \vee \cdots \vee p_k) \rightarrow q) \leftrightarrow ((p_1 \rightarrow q) \wedge \cdots \wedge (p_k \rightarrow q))$$

- Each of the implications $p_i \rightarrow q$ is a case

# If $n$ is an integer then $n^2 \geq n$

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Constructive proof: exhibit an actual witness $w$ from the domain such that $P(w)$ is true. Therefore, $\exists x\ P(x)$

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- 1729 is such a number because
- $10^3 + 9^3 = 1729 = 12^3 + 1^3$

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- Show that there must be a value $v$ such that $P(v)$ is true
- But we don't know what this value $v$ is

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Proof We need only prove the existence of at least one example. Consider the case $x = \sqrt{2}$ and $y = \sqrt{2}$. We distinguish two cases:

Case $\sqrt{2}^{\sqrt{2}}$ is rational. In that case we have shown that for the irrational numbers $x = y = \sqrt{2}$, we have that $x^y$ is rational

Case $\sqrt{2}^{\sqrt{2}}$ is irrational. In that case consider $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. We then have that

$$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$$

But since 2 is rational, we have shown that for $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, we have that $x^y$ is rational

We have thus shown that in any case there exist some irrational numbers $x$ and $y$ such that $x^y$ is rational

# Disproving $\forall x\ P(x)$ with a counter-example

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- To establish that $\neg\forall x\ P(x)$ is true find a $w$ such that $P(w)$ is false
- So, $w$ is a counterexample to the assertion $\forall x\ P(x)$

# Every positive integer is the sum of the squares of 3 integers

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The integer 7 is a counterexample. So the claim is false

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$$\forall x \; (x \neq 0 \;\rightarrow\; \exists y \; (x \times y = 1))$$

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$$\forall x\ (x \neq 0\ \rightarrow\ \exists y\ (x \times y = 1))$$

- $\lim_{x \rightarrow a} f(x) = b$

$$\forall \epsilon\ \exists \delta\ \forall x\ (0 < |x - a| < \delta \rightarrow |f(x) - b| < \epsilon)$$

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- Every real number has an inverse w.r.t addition (domain $\mathbb{R}$)

$$\forall x\ \exists y\ (x + y = 0)$$

- Every real number except zero has an inverse w.r.t multiplication

$$\forall x\ (x \neq 0\ \rightarrow\ \exists y\ (x \times y = 1))$$

- $\lim_{x \to a} f(x) = b$

$$\forall \epsilon\ \exists \delta\ \forall x\ (0 < |x - a| < \delta \rightarrow |f(x) - b| < \epsilon)$$

- $\neg(\lim_{x \to a} f(x) = b)$

$$\exists \epsilon\ \forall \delta\ \exists x\ ((0 < |x - a| < \delta) \wedge (|f(x) - b| \geq \epsilon))$$