Discrete Mathematics & Mathematical Reasoning
Predicates, Quantifiers and Proof Techniques

Colin Stirling
Informatics
Recall propositional logic from last year (in Inf1CL)

Propositions can be constructed from other propositions using logical connectives

Negation: \( \neg \)

Conjunction: \( \land \)

Disjunction: \( \lor \)

Implication: \( \rightarrow \)

Biconditional: \( \leftrightarrow \)
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Propositions can be constructed from other propositions using logical connectives

- Negation: \( \neg \)
- Conjunction: \( \wedge \)
- Disjunction: \( \vee \)
- Implication: \( \rightarrow \)
- Biconditional: \( \leftrightarrow \)

The truth of a proposition is defined by the truth values of its elementary propositions and the meaning of connectives
Recall propositional logic from last year (in Inf1CL)

Propositions can be constructed from other propositions using logical connectives

- Negation: $\neg$
- Conjunction: $\land$
- Disjunction: $\lor$
- Implication: $\rightarrow$
- Biconditional: $\leftrightarrow$

The truth of a proposition is defined by the truth values of its elementary propositions and the meaning of connectives

The meaning of logical connectives can be defined using truth tables
Examples of logical implication and equivalence

- \((p \land (p \rightarrow q)) \rightarrow q\)
- \((p \land \neg p) \rightarrow q\)
- \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)

\(\vdots\)
Examples of logical implication and equivalence

- \((p \land (p \to q)) \to q\)
- \((p \land \neg p) \to q\)
- \(((p \to q) \land (q \to r)) \to (p \to r)\)
- ...
- \((p \to q) \iff (\neg q \to \neg p)\)
  Contraposition
- \(\neg(p \land q) \iff (\neg p \lor \neg q)\)
  De Morgan
- \(\neg(p \lor q) \iff (\neg p \land \neg q)\)
  De Morgan
- \(\neg(p \to q) \iff (p \land \neg q)\)
- ...
- ...
Propositional logic is not “enough”
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In propositional logic, from

- All cats have whiskers (proposition $p$)
- Sansa is a cat (proposition $q$)

we cannot derive

- Sansa has whiskers (proposition $r$)
Propositional logic is not “enough”

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- All cats have whiskers (proposition $p$)
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- Sansa has whiskers (proposition $r$)
- $(p \land q) \rightarrow r$ is not a tautology
Propositional logic is not “enough”

In propositional logic, from

- All cats have whiskers \((\text{proposition } p)\)
- Sansa is a cat \((\text{proposition } q)\)

we cannot derive

- Sansa has whiskers \((\text{proposition } r)\)

\((p \land q) \rightarrow r\) is not a tautology

We need a language to talk about objects, their properties and their relations
Sansa the cat (with whiskers)
Formally same argument as

Given the following two premises

- All students in this class understand logic
- Colin is a student in this class
Formally same argument as

Given the following two premises

- All students in this class understand logic
- Colin is a student in this class

it follows that

- Colin understands logic
Predicate logic

Extends propositional logic by the new features

- Variables: $x, y, z, \ldots$
- Predicates: $P(x), Q(x), R(x, y), M(x, y, z), \ldots$
- Quantifiers: $\forall, \exists$

Predicates are a generalisation of propositions
Can contain variables
Variables stand for (and can be replaced by) elements from their domain
The truth value of a predicate depends on the values of its variables
Predicate logic

Extends propositional logic by the new features

- Variables: \( x, y, z, \ldots \)
- Predicates: \( P(x), Q(x), R(x, y), M(x, y, z), \ldots \)
- Quantifiers: \( \forall, \exists \)

Predicates are a generalisation of propositions

- Can contain variables \( M(x, y, z) \)
- Variables stand for (and can be replaced by) elements from their domain
- The truth value of a predicate depends on the values of its variables
Examples

$P(x)$ is “$x > 5$” and $x$ ranges over $\mathbb{Z}$ (integers)

- $P(8)$ is true
- $P(-1)$ is false
Examples

\( P(x) \) is “\( x > 5 \)” and \( x \) ranges over \( \mathbb{Z} \) (integers)
- \( P(8) \) is true
- \( P(-1) \) is false

\( C(x) \) is “\( x \) is a cat”; \( W(x) \) is “\( x \) has whiskers” and \( x \) ranges over animals
- \( C(Sansa) \) is true
- \( C(Colin) \) is false
- \( W(Sansa) \) is true
Examples

\(P(x)\) is “\(x > 5\)” and \(x\) ranges over \(\mathbb{Z}\) (integers)

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- \(C(Sansa)\) is true
- \(C(Colin)\) is false
- \(W(Sansa)\) is true

\(D(x, y)\) is “\(x\) divides \(y\)” and \(x, y\) range over \(\mathbb{Z}^+\) (positive integers)

- \(D(3, 9)\) is true
- \(D(2, 9)\) is false
Examples

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- \( C(Sansa) \) is true
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\( D(x, y) \) is “\( x \) divides \( y \)” and \( x, y \) range over \( \mathbb{Z}^+ \) (positive integers)
- \( D(3, 9) \) is true
- \( D(2, 9) \) is false

\( S(x_1, \ldots, x_{11}, y) \) is “\( x_1 + \ldots + x_{11} = y \)”
- \( S(1, 2, \ldots, 11, 66) \) is true
Quantifiers

- **Universal quantifier, “For all”:** \( \forall \)
  \[ \forall x \ P(x) \] asserts that \( P(x) \) is true for every \( x \) in the assumed domain.

- **Existential quantifier, “There exists”:** \( \exists \)
  \[ \exists x \ P(x) \] asserts that \( P(x) \) is true for some \( x \) in the assumed domain.

The quantifiers are said to bind the variable \( x \) in these expressions. Variables in the scope of some quantifier are called **bound variables**. All other variables in the expression are called **free variables**. A formula that does not contain any free variables is a proposition.
Quantifiers

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Quantifiers

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- The quantifiers are said to bind the variable \( x \) in these expressions. Variables in the scope of some quantifier are called bound variables. All other variables in the expression are called free variables.

- A formula that does not contain any free variables is a proposition and has a truth value.
Quantifier Rule

- Rule of inference

\[
\begin{array}{c}
\forall x \ P(x) \\
\hline
P(v)
\end{array}
\]

\(v\) is a value in assumed domain

\(\neg (\forall x \ P(x)) \iff \exists x \ \neg P(x)\)
\(\neg (\exists x \ P(x)) \iff \forall x \ \neg P(x)\)

It is not the case that for all \(x\) \(P(x)\) if, and only if, \(P(x)\) is not true for some \(x\)

We always assume that a domain is nonempty
Quantifier Rule

- Rule of inference

\[
\frac{\forall x \ P(x)}{P(v)} \quad \text{\(v\) is a value in assumed domain}
\]

From \(\forall x \ P(x)\) is true infer that \(P(v)\) is true for any value \(v\) in the assumed domain

- \(\neg(\forall x \ P(x)) \leftrightarrow \exists x \ \neg P(x)\) \(\neg(\exists x \ P(x)) \leftrightarrow \forall x \ \neg P(x)\)
Quantifier Rule

- Rule of inference

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- \(\neg(\forall x \ P(x)) \iff \exists x \ \neg P(x) \quad \neg(\exists x \ P(x)) \iff \forall x \ \neg P(x)\)

It is not the case that for all \(x\) \(P(x)\) if, and only if, \(P(x)\) is not true for some \(x\)
Quantifier Rule

- Rule of inference

\[
\frac{\forall x \ P(x)}{P(v)} \quad v \text{ is a value in assumed domain}
\]

From \(\forall x \ P(x)\) is true infer that \(P(v)\) is true for any value \(v\) in the assumed domain

- \(\neg(\forall x \ P(x)) \iff \exists x \ \neg P(x)\) \quad \(\neg(\exists x \ P(x)) \iff \forall x \ \neg P(x)\)

It is not the case that for all \(x\) \(P(x)\) if, and only if, \(P(x)\) is not true for some \(x\)

- We always assume that a domain is nonempty
Our example

- From *All cats have whiskers and Sansa is a cat* derive *Sansa has whiskers*
Our example

- From All cats have whiskers and Sansa is a cat derive Sansa has whiskers
- $C(x)$ is “$x$ is a cat”; $W(x)$ is “$x$ has whiskers”
Our example

- From All cats have whiskers and Sansa is a cat derive Sansa has whiskers
- $C(x)$ is “$x$ is a cat”; $W(x)$ is “$x$ has whiskers”
- All cats have whiskers $\forall x \ (C(x) \to W(x))$
Our example

- From **All cats have whiskers** and **Sansa is a cat** derive **Sansa has whiskers**
- $C(x)$ is "$x$ is a cat"; $W(x)$ is "$x$ has whiskers"
- All cats have whiskers $\forall x (C(x) \rightarrow W(x))$
- Sansa is a cat $C(Sansa)$
Our example

- From **All cats have whiskers** and **Sansa is a cat** derive **Sansa has whiskers**
- $C(x)$ is “$x$ is a cat”; $W(x)$ is “$x$ has whiskers”
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- Sansa is a cat $C(Sansa)$
- How do we get $W(Sansa)$?
Our example

- From All cats have whiskers and Sansa is a cat derive Sansa has whiskers
- $C(x)$ is “$x$ is a cat”; $W(x)$ is “$x$ has whiskers”
- All cats have whiskers  $\forall x \ (C(x) \rightarrow W(x))$
- Sansa is a cat  $C(Sansa)$
- How do we get $W(Sansa)$?
- From $\forall x \ (C(x) \rightarrow W(x))$ we derive $C(Sansa) \rightarrow W(Sansa)$
Our example

- From **All cats have whiskers** and **Sansa is a cat** derive **Sansa has whiskers**
- $C(x)$ is “$x$ is a cat”; $W(x)$ is “$x$ has whiskers”
- All cats have whiskers: $\forall x (C(x) \rightarrow W(x))$
- Sansa is a cat: $C(\text{Sansa})$
- How do we get $W(\text{Sansa})$?
- From $\forall x (C(x) \rightarrow W(x))$ we derive $C(\text{Sansa}) \rightarrow W(\text{Sansa})$
- By propositional reasoning, $(p \rightarrow q$ and $p)$ implies $q$
Our example

- From **All cats have whiskers** and **Sansa is a cat** derive **Sansa has whiskers**
- \( C(x) \) is “\( x \) is a cat”; \( W(x) \) is “\( x \) has whiskers”
- All cats have whiskers \( \forall x \ (C(x) \rightarrow W(x)) \)
- Sansa is a cat \( C(\text{Sansa}) \)
- How do we get \( W(\text{Sansa}) \)?
- From \( \forall x \ (C(x) \rightarrow W(x)) \) we derive \( C(\text{Sansa}) \rightarrow W(\text{Sansa}) \)
- By propositional reasoning, \( (p \rightarrow q \text{ and } p) \) implies \( q \)
  - So, \( (C(\text{Sansa}) \rightarrow W(\text{Sansa}) \text{ and } C(\text{Sansa})) \) implies \( W(\text{Sansa}) \)
Another example

Given the following three premises

- All hummingbirds are richly coloured
- No large birds live on honey
- Birds that do not live on honey are dull in colour
Another example

Given the following three premises

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- No large birds live on honey
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Does it follow that

- Hummingbirds are small
Another example

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Does it follow that

- Hummingbirds are small

Uses derived rule of transitivity: from

- \( \forall x (A(x) \rightarrow B(x)) \)
- \( \forall x (B(x) \rightarrow C(x)) \)
Another example

Given the following three premises

- All hummingbirds are richly coloured
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Does it follow that

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Uses derived rule of transitivity: from

\[ \forall x (A(x) \rightarrow B(x)) \]
\[ \forall x (B(x) \rightarrow C(x)) \]
\[ \forall x (A(x) \rightarrow C(x)) \] follows
Proving $\forall x \ P(x)$

- Rule of inference

\[
\frac{P(c)}{\forall x \ P(x)} \text{ c is an arbitrary element of domain}
\]
Proving $\forall x \ P(x)$

- Rule of inference

\[
\begin{align*}
\frac{P(c)}{\forall x \ P(x)} & \quad c \text{ is an arbitrary element of domain}
\end{align*}
\]

- Derived rule of transitivity (using transitivity of implication)
Proving $\forall x \ P(x)$

- Rule of inference
  
  $\frac{P(c)}{\forall x \ P(x)} \quad c$ is an arbitrary element of domain

- Derived rule of transitivity (using transitivity of implication)

- Example: if $n$ is an odd integer then $n^2$ is odd
Proving $\forall x \ P(x)$

- Rule of inference

$$\frac{P(c)}{\forall x \ P(x)} \quad c \text{ is an arbitrary element of domain}$$

- Derived rule of transitivity (using transitivity of implication)
- Example: if $n$ is an odd integer then $n^2$ is odd
- First, notice the quantifier is implicit
Proving $\forall x \ P(x)$

- Rule of inference
  \[
  \frac{P(c)}{\forall x \ P(x)} \quad c \text{ is an arbitrary element of domain}
  \]

- Derived rule of transitivity (using transitivity of implication)
- Example: if $n$ is an odd integer then $n^2$ is odd
- First, notice the quantifier is implicit
- Let $P(n)$ be “$n$ is odd” and $Q(n)$ be “the square of $n$ is odd”
Proving $\forall x \ P(x)$

- Rule of inference
  \[
  \frac{P(c)}{\forall x \ P(x)} \quad c \text{ is an arbitrary element of domain}
  \]

- Derived rule of transitivity (using transitivity of implication)
- Example: if $n$ is an odd integer then $n^2$ is odd
- First, notice the quantifier is implicit

Let $P(n)$ be “$n$ is odd” and $Q(n)$ be “the square of $n$ is odd”

So is: $\forall x \ (P(x) \rightarrow Q(x))$ where domain is integers
Direct proof of $\forall x (P(x) \rightarrow Q(x))$

1. Assume $n$ is an arbitrary element of the domain
Direct proof of $\forall x (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
- Use the definition/properties of $P(n)$, $n$ is odd
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
- Use the definition/properties of $P(n)$, $n$ is odd
- $P(n)$ provided that for some $k$, $n = 2k + 1$
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
- Use the definition/properties of $P(n)$, $n$ is odd
- $P(n)$ provided that for some $k$, $n = 2k + 1$
- So $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
- Use the definition/properties of $P(n)$, $n$ is odd
- $P(n)$ provided that for some $k$, $n = 2k + 1$
- So $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$
- $n^2$ has the form for some $m$, $n^2 = 2m + 1$; so $Q(n)$
Any odd integer is the difference of two squares
Proving $\forall x \ (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$

Assume $c$ is an arbitrary element of the domain

Prove that $\neg B(c) \rightarrow \neg A(c)$

That is, assume $\neg B(c)$ then show $\neg A(c)$

Use the definition/properties of $\neg B(c)$
Proving $\forall x (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$
- So, $\forall x (A(x) \rightarrow B(x)) \iff \forall x (\neg B(x) \rightarrow \neg A(x))$
Proving $\forall x (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$
- So, $\forall x (A(x) \rightarrow B(x)) \leftrightarrow \forall x (\neg B(x) \rightarrow \neg A(x))$
- Assume $c$ is an arbitrary element of the domain
Proving $\forall x (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$
- So, $\forall x (A(x) \rightarrow B(x)) \iff \forall x (\neg B(x) \rightarrow \neg A(x))$
- Assume $c$ is an arbitrary element of the domain
- Prove that $\neg B(c) \rightarrow \neg A(c)$
Proving \( \forall x (A(x) \rightarrow B(x)) \) by contraposition

- Uses equivalence of \((p \rightarrow q) \) and \((\neg q \rightarrow \neg p)\)
- So, \( \forall x (A(x) \rightarrow B(x)) \Leftrightarrow \forall x (\neg B(x) \rightarrow \neg A(x)) \)
- Assume \( c \) is an arbitrary element of the domain
- Prove that \( \neg B(c) \rightarrow \neg A(c) \)
- That is, assume \( \neg B(c) \) then show \( \neg A(c) \)
Proving $\forall x (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$
- So, $\forall x (A(x) \rightarrow B(x)) \leftrightarrow \forall x (\neg B(x) \rightarrow \neg A(x))$
- Assume $c$ is an arbitrary element of the domain
- Prove that $\neg B(c) \rightarrow \neg A(c)$
- That is, assume $\neg B(c)$ then show $\neg A(c)$
- Use the definition/properties of $\neg B(c)$
if $x + y$ is even, then $x$ and $y$ have the same parity

Proof Let $n, m \in \mathbb{Z}$ be arbitrary. We will prove that if $n$ and $m$ do not have the same parity then $n + m$ is odd. Without loss of generality we assume that $n$ is odd and $m$ is even, that is $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $m = 2\ell$ for some $\ell \in \mathbb{Z}$. But then $n + m = 2k + 1 + 2\ell = 2(k + \ell) + 1$. And thus $n + m$ is odd. Now by equivalence of a statement with it contrapositive derive that if $n + m$ is even, then $n$ and $m$ have the same parity.
if $x + y$ is even, then $x$ and $y$ have the same parity

Proof Let $n, m \in \mathbb{Z}$ be arbitrary. We will prove that if $n$ and $m$ do not have the same parity then $n + m$ is odd. Without loss of generality we assume that $n$ is odd and $m$ is even, that is $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $m = 2\ell$ for some $\ell \in \mathbb{Z}$. But then

$$n + m = 2k + 1 + 2\ell = 2(k + \ell) + 1.$$ And thus $n + m$ is odd. Now by equivalence of a statement with it contrapositive derive that if $n + m$ is even, then $n$ and $m$ have the same parity.
If $n = ab$ where $a, b$ are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
Proof by contradiction

Want to prove that $p$ is true
Proof by contradiction

- Want to prove that $p$ is true
- Assume $\neg p$
Proof by contradiction

- Want to prove that $p$ is true
- Assume $\neg p$
- Derive both $q$ and $\neg q$ for some $q$ (a contradiction equivalent to False)

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Proof by contradiction

- Want to prove that $p$ is true
- Assume $\neg p$
- Derive both $q$ and $\neg q$ for some $q$ (a contradiction equivalent to False)
- Therefore, $\neg \neg p$ which is equivalent to $p$
\[ \sqrt{2} \text{ is irrational} \]

Proof: Assume towards a contradiction that \( \sqrt{2} \) is rational, that is there are integers \( a \) and \( b \) with no common factor other than 1, such that \( \sqrt{2} = \frac{a}{b} \). In that case \( 2 = \frac{a^2}{b^2} \). Multiplying both sides by \( b^2 \), we have \( a^2 = 2b^2 \). Since \( b \) is an integer, so is \( b^2 \), and thus \( a^2 \) is even. As we saw previously this implies that \( a \) is even, that is there is an integer \( c \) such that \( a = 2c \). Hence \( 2b^2 = 4c^2 \), hence \( b^2 = 2c^2 \). Now, since \( c \) is an integer, so is \( c^2 \), and thus \( b^2 \) is even. Again, we can conclude that \( b \) is even. Thus \( a \) and \( b \) have a common factor 2, contradicting the assertion that \( a \) and \( b \) have no common factor other than 1. This shows that the original assumption that \( \sqrt{2} \) is rational is false, and that \( \sqrt{2} \) must be irrational.
Proof Assume towards a contradiction that $\sqrt{2}$ is rational, that is there are integers $a$ and $b$ with no common factor other than 1, such that $\sqrt{2} = a/b$. In that case $2 = a^2/b^2$. Multiplying both sides by $b^2$, we have $a^2 = 2b^2$. Since $b$ is an integer, so is $b^2$, and thus $a^2$ is even. As we saw previously this implies that $a$ is even, that is there is an integer $c$ such that $a = 2c$. Hence $2b^2 = 4c^2$, hence $b^2 = 2c^2$. Now, since $c$ is an integer, so is $c^2$, and thus $b^2$ is even. Again, we can conclude that $b$ is even. Thus $a$ and $b$ have a common factor 2, contradicting the assertion that $a$ and $b$ have no common factor other than 1. This shows that the original assumption that $\sqrt{2}$ is rational is false, and that $\sqrt{2}$ must be irrational.
There are infinitely many primes

Lemma: Every natural number greater than one is either prime or it has a prime divisor.

Proof: Suppose towards a contradiction that there are only finitely many primes $p_1, p_2, p_3, \ldots, p_k$. Consider the number $q = p_1 p_2 p_3 \ldots p_k + 1$, the product of all the primes plus one. By hypothesis $q$ cannot be prime because it is strictly larger than all the primes. Thus, by the lemma, it has a prime divisor, $p$. Because $p_1, p_2, p_3, \ldots, p_k$ are all the primes, $p$ must be equal to one of them, so $p$ is a divisor of their product. So we have that $p$ divides $p_1 p_2 p_3 \ldots p_k$, and $p$ divides $q$, but that means $p$ divides their difference, which is 1. Therefore $p \leq 1$. Contradiction. Therefore there are infinitely many primes.
There are infinitely many primes

Lemma Every natural number greater than one is either prime or it has a prime divisor
There are infinitely many primes

Lemma Every natural number greater than one is either prime or it has a prime divisor

Proof Suppose towards a contradiction that there are only finitely many primes $p_1, p_2, p_3, \ldots, p_k$. Consider the number $q = p_1 p_2 p_3 \ldots p_k + 1$, the product of all the primes plus one. By hypothesis $q$ cannot be prime because it is strictly larger than all the primes. Thus, by the lemma, it has a prime divisor, $p$. Because $p_1, p_2, p_3, \ldots, p_k$ are all the primes, $p$ must be equal to one of them, so $p$ is a divisor of their product. So we have that $p$ divides $p_1 p_2 p_3 \ldots p_k$, and $p$ divides $q$, but that means $p$ divides their difference, which is 1. Therefore $p \leq 1$. Contradiction. Therefore there are infinitely many primes.
Proof by cases

- To prove a conditional statement of the form

\[(p_1 \lor \cdots \lor p_k) \rightarrow q\]

- Use the tautology

\[((p_1 \lor \cdots \lor p_k) \rightarrow q) \iff ((p_1 \rightarrow q) \land \cdots \land (p_k \rightarrow q))\]

- Each of the implications \(p_i \rightarrow q\) is a case
If $n$ is an integer then $n^2 \geq n$
Proof of $\exists x \ P(x)$

Rule of inference

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Constructive proof: exhibit an actual witness $w$ from the domain such that $P(w)$ is true. Therefore, $\exists x \ P(x)$
There exists a positive integer that can be written as the sum of cubes of positive integers in two different ways

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Nonconstructive proof of $\exists x \ P(x)$

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But we don't know what this value $v$ is.
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There exist irrational numbers $x$ and $y$ such that $x^y$ is rational.

Proof

We need only prove the existence of at least one example. Consider the case $x = \sqrt{2}$ and $y = \sqrt{2}$. We distinguish two cases:

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. In that case we have shown that for the irrational numbers $x = y = \sqrt{2}$, we have that $x^y$ is rational.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. In that case consider $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. We then have that $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$. But since 2 is rational, we have shown that for $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, we have that $x^y$ is rational.

We have thus shown that in any case there exist some irrational numbers $x$ and $y$ such that $x^y$ is rational.
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Disproving $\forall x \ P(x)$ with a counter-example

- $\neg \forall x \ P(x)$ is equivalent to $\exists x \ \neg P(x)$

To establish that $\neg \forall x \ P(x)$ is true find a $w$ such that $P(w)$ is false.

So, $w$ is a counterexample to the assertion $\forall x \ P(x)$. 
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Every positive integer is the sum of the squares of 3 integers
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The integer 7 is a counterexample. So the claim is false
Nested quantifiers

- Every real number has an inverse w.r.t addition (domain $\mathbb{R}$)

$$\forall x \exists y \ (x + y = 0)$$
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- Every real number except zero has an inverse w.r.t multiplication
  \[ \forall x \ (x \neq 0 \rightarrow \exists y \ (x \times y = 1)) \]
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  \[ \forall \epsilon \ \exists \delta \ \forall x \ (0 < |x - a| < \delta \rightarrow |f(x) - b| < \epsilon) \]
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- $\neg (\lim_{x \to a} f(x) = b)$

$$\exists \varepsilon \ \forall \delta \ \exists x \ ((0 < |x - a| < \delta) \ \land \ (|f(x) - b| \geq \varepsilon))$$