# Discrete Mathematics & Mathematical Reasoning
# Multiplicative Inverses and Some Cryptography

Colin Stirling

Informatics

# Multiplicative inverses

- Every real number $x$, except $x = 0$, has a multiplicative inverse $y = \frac{1}{x}$; so $xy = 1$

# Multiplicative inverses

- Every real number $x$, except $x = 0$, has a multiplicative inverse $y = \frac{1}{x}$; so $xy = 1$
- Similarly for $x$ mod $m$, except $x = 0$, we wish to find $y$ mod $m$ such that $xy \equiv 1 \pmod{m}$

# Multiplicative inverses

- Every real number $x$, except $x = 0$, has a multiplicative inverse $y = \frac{1}{x}$; so $xy = 1$
- Similarly for $x$ mod $m$, except $x = 0$, we wish to find $y$ mod $m$ such that $xy \equiv 1 \pmod{m}$
- $x = 8$ and $m = 15$. Then $x\,2 = 16 \equiv 1 \pmod{15}$, so 2 is a multiplicative inverse of 8 (mod 15)

# Multiplicative inverses

- Every real number $x$, except $x = 0$, has a multiplicative inverse $y = \frac{1}{x}$; so $xy = 1$
- Similarly for $x$ mod $m$, except $x = 0$, we wish to find $y$ mod $m$ such that $xy \equiv 1 \pmod{m}$
- $x = 8$ and $m = 15$. Then $x\ 2 = 16 \equiv 1 \pmod{15}$, so 2 is a multiplicative inverse of 8 (mod 15)
- $x = 12$ and $m = 15$

# Multiplicative inverses

- Every real number $x$, except $x = 0$, has a multiplicative inverse $y = \frac{1}{x}$; so $xy = 1$
- Similarly for $x$ mod $m$, except $x = 0$, we wish to find $y$ mod $m$ such that $xy \equiv 1 \pmod{m}$
- $x = 8$ and $m = 15$. Then $x\ 2 = 16 \equiv 1 \pmod{15}$, so 2 is a multiplicative inverse of 8 (mod 15)
- $x = 12$ and $m = 15$
  The sequence $\{xa \pmod{m} \mid a = 0, 1, 2, ...\}$ is periodic, and takes on the values $\{0, 12, 9, 6, 3\}$. So, 12 has no multiplicative inverse mod 15

# Multiplicative inverses

- Every real number $x$, except $x = 0$, has a multiplicative inverse $y = \frac{1}{x}$; so $xy = 1$
- Similarly for $x$ mod $m$, except $x = 0$, we wish to find $y$ mod $m$ such that $xy \equiv 1 \pmod{m}$
- $x = 8$ and $m = 15$. Then $x\,2 = 16 \equiv 1 \pmod{15}$, so 2 is a multiplicative inverse of 8 (mod 15)
- $x = 12$ and $m = 15$
  The sequence $\{xa \pmod{m} \mid a = 0, 1, 2, ...\}$ is periodic, and takes on the values $\{0, 12, 9, 6, 3\}$. So, 12 has no multiplicative inverse mod 15
- Notice $\gcd(8, 15) = 1$ whereas $\gcd(12, 15) = 3$

# Multiplicative inverses mod $m$ when $\gcd(m, x) = 1$

**Theorem**

*If $m, x$ are positive integers and $\gcd(m, x) = 1$ then $x$ has a multiplicative inverse mod m (and it is unique mod m)*

# Multiplicative inverses mod $m$ when $\gcd(m, x) = 1$

**Theorem**

*If $m, x$ are positive integers and $\gcd(m, x) = 1$ then $x$ has a multiplicative inverse mod $m$ (and it is unique mod $m$)*

**Proof.**

By Bézout's theorem there are $s$ and $t$ such that

$$sm + tx = 1 = \gcd(m, x)$$

So, $sm + tx \equiv 1 \pmod{m}$. As $sm \equiv 0 \pmod{m}$, so $tx \equiv 1 (\text{mod } m)$. For uniqueness mod $m$. Assume $tx \equiv 1 \pmod{m}$ and $ux \equiv 1 \pmod{m}$. Therefore, $tx \equiv ux \pmod{m}$. Since $\gcd(m, x) = 1$ it follows that $t \equiv u \pmod{m}$. □

# Multiplicative inverses mod $m$ when $\gcd(m, x) = 1$

**Theorem**

*If $m, x$ are positive integers and $\gcd(m, x) = 1$ then $x$ has a multiplicative inverse mod m (and it is unique mod m)*

**Proof.**

By Bézout's theorem there are $s$ and $t$ such that

$$sm + tx = 1 = \gcd(m, x)$$

So, $sm + tx \equiv 1 \pmod{m}$. As $sm \equiv 0 \pmod{m}$, so $tx \equiv 1 (\text{mod } m)$. For uniqueness mod $m$. Assume $tx \equiv 1 \pmod{m}$ and $ux \equiv 1 \pmod{m}$. Therefore, $tx \equiv ux \pmod{m}$. Since $\gcd(m, x) = 1$ it follows that $t \equiv u \pmod{m}$. □

Compute the multiplicative inverse using extended euclidean algorithm

# Chinese remainder theorem

## Theorem

*Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ be arbitrary integers. Then the system*

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_n \pmod{m_n} \end{aligned}$$

*has a unique solution modulo $m = m_1 m_2 \cdots m_n$*

# Chinese remainder theorem

## Theorem

*Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ be arbitrary integers. Then the system*

$$\begin{aligned} x &\equiv a_1 \ (mod\ m_1) \\ x &\equiv a_2 \ (mod\ m_2) \\ &\vdots \\ x &\equiv a_n \ (mod\ m_n) \end{aligned}$$

*has a unique solution modulo $m = m_1 m_2 \cdots m_n$*

## Proof.

In the book □

# Example

$$
\begin{aligned}
x &\equiv 2 \pmod{3} \\
x &\equiv 3 \pmod{5} \\
x &\equiv 5 \pmod{7}
\end{aligned}
$$

# Example

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 5 \pmod{7}$$

- $m = 3 \cdot 5 \cdot 7 = 105$

# Example

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 5 \\ x &\equiv 5 \pmod 7 \end{aligned}$$

- $m = 3 \cdot 5 \cdot 7 = 105$
- $M_1 = 35$ and 2 is an inverse of $M_1$ mod 3

# Example

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 5 \\ x &\equiv 5 \pmod 7 \end{aligned}$$

- $m = 3 \cdot 5 \cdot 7 = 105$
- $M_1 = 35$ and 2 is an inverse of $M_1$ mod 3
- $M_2 = 21$ and 1 is an inverse of $M_2$ mod 5

# Example

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 5 \pmod 7$$

- $m = 3 \cdot 5 \cdot 7 = 105$
- $M_1 = 35$ and 2 is an inverse of $M_1$ mod 3
- $M_2 = 21$ and 1 is an inverse of $M_2$ mod 5
- $M_3 = 15$ and 1 is an inverse of $M_3$ mod 7

# Example

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 5 \pmod 7$$

- $m = 3 \cdot 5 \cdot 7 = 105$
- $M_1 = 35$ and 2 is an inverse of $M_1$ mod 3
- $M_2 = 21$ and 1 is an inverse of $M_2$ mod 5
- $M_3 = 15$ and 1 is an inverse of $M_3$ mod 7
- $x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 5 \cdot 15 \cdot 1$

# Example

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 5 \pmod 7$$

- $m = 3 \cdot 5 \cdot 7 = 105$
- $M_1 = 35$ and 2 is an inverse of $M_1$ mod 3
- $M_2 = 21$ and 1 is an inverse of $M_2$ mod 5
- $M_3 = 15$ and 1 is an inverse of $M_3$ mod 7
- $x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 5 \cdot 15 \cdot 1$
- $x = 140 + 63 + 75 = 278 \equiv 68 \pmod{105}$

# Fermat's little theorem

## Theorem

*If p is prime and $p \not| a$, then $a^{p-1} \equiv 1 \pmod{p}$. Furthermore, for every integer a we have $a^p \equiv a \pmod{p}$*

# Fermat's little theorem

## Theorem

*If $p$ is prime and $p \not| a$, then $a^{p-1} \equiv 1 \pmod{p}$. Furthermore, for every integer $a$ we have $a^p \equiv a \pmod{p}$*

## Proof.

Assume $p \not| a$ and so, therefore, $\gcd(p, a) = 1$. Then $a, 2a, \ldots, (p-1)a$ are not pairwise congruent modulo $p$; if $ia \equiv ja \pmod{p}$ because $\gcd(p, a) = 1$ then $i \equiv j \pmod{p}$ which is impossible. Therefore, each element $ja \bmod p$ is a distinct element in the set $\{1, \ldots, p-1\}$. This means that the product $a \cdot 2a \cdots (p-1)a \equiv 1 \cdot 2 \cdots p-1 \pmod{p}$. Therefore, $(p-1)!a^{p-1} \equiv (p-1)! \pmod{p}$. Now because $\gcd(p, q) = 1$ for $1 \leq q \leq p-1$ it follows that $a^{p-1} \equiv 1 \pmod{p}$. Therefore, also $a^p \equiv a \pmod{p}$ and when $p|a$ then clearly $a^p \equiv a \pmod{p}$. □

# Computing the remainders modulo prime $p$

- Find $7^{222}$ mod 11

# Computing the remainders modulo prime $p$

- Find $7^{222}$ mod 11
- By Fermat's little theorem, we know that $7^{10} \equiv 1 \pmod{11}$, and so $(7^{10})^k \equiv 1 \pmod{11}$ for every positive integer $k$. Therefore, $7^{222} = 7^{22 \cdot 10+2} = (7^{10})^{22}\, 7^2 \equiv 1^{22} 49 \equiv 5 \pmod{11}$. Hence, $7^{222} \bmod 11 = 5$

# Computing the remainders modulo prime $p$

- Find $7^{222} \bmod 11$
- By Fermat's little theorem, we know that $7^{10} \equiv 1 \pmod{11}$, and so $(7^{10})^k \equiv 1 \pmod{11}$ for every positive integer $k$. Therefore, $7^{222} = 7^{22 \cdot 10 + 2} = (7^{10})^{22}\, 7^2 \equiv 1^{22} 49 \equiv 5 \pmod{11}$. Hence, $7^{222} \bmod 11 = 5$
- $2^{340} \equiv 1 \pmod{11}$ because $2^{10} \equiv 1 \pmod{11}$

# Private key cryptography

- Bob wants to send Alice a secret message M

# Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)

# Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)
- Bob encrypts M and sends Alice En(M)

# Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))

# Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))
- Important property De(En(M)) $=$ M

# Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))
- Important property De(En(M)) $=$ M
- Alice and Bob share a secret which could be intercepted by a third party

# Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))
- Important property De(En(M)) $=$ M
- Alice and Bob share a secret which could be intercepted by a third party
- Example use En($p$) $= (p + 3) \bmod 26$

# Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))
- Important property De(En(M)) $=$ M
- Alice and Bob share a secret which could be intercepted by a third party
- Example use En($p$) $= (p + 3)$ mod 26
- What is WKLV LV D VHFUHW ?

# Public key cryptography

- Bob wants to send Alice a secret message M

# Public key cryptography

- Bob wants to send Alice a secret message M
- Without Alice and Bob sharing a secret

# Public key cryptography

- Bob wants to send Alice a secret message M
- Without Alice and Bob sharing a secret
- Alice sends Bob a public key En (and keeps her inverse private key De secret from everyone including Bob)

# Public key cryptography

- Bob wants to send Alice a secret message M
- Without Alice and Bob sharing a secret
- Alice sends Bob a public key En (and keeps her inverse private key De secret from everyone including Bob)
- Bob encrypts M and sends Alice En(M)

# Public key cryptography

- Bob wants to send Alice a secret message M
- Without Alice and Bob sharing a secret
- Alice sends Bob a public key En (and keeps her inverse private key De secret from everyone including Bob)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))

# Public key cryptography

- Bob wants to send Alice a secret message M
- Without Alice and Bob sharing a secret
- Alice sends Bob a public key En (and keeps her inverse private key De secret from everyone including Bob)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))
- Important property De(En(M)) $=$ M

# Public key cryptography

- Bob wants to send Alice a secret message M
- Without Alice and Bob sharing a secret
- Alice sends Bob a public key En (and keeps her inverse private key De secret from everyone including Bob)
- Bob encrypts M and sends Alice En(M)
- Alice decrypts En(M), De(En(M))
- Important property De(En(M)) $=$ M
- The challenge: De can't be feasibly computed from En; and given En(M) one can't feasibly compute M

# RSA Cryptosystem: Rivest, Shamir and Adleman

- Choose two distinct prime numbers $p$ and $q$
- Let $n = pq$ and $k = (p-1)(q-1)$
- Choose integer $e$ where $1 < e < k$ and $\gcd(e, k) = 1$
- $(n, e)$ is released as the public key
- Let $d$ be the multiplicative inverse of $e$ modulo $k$, so $de \equiv 1 \pmod{k}$
- $(n, d)$ is the private key and kept secret

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

**Encryption** Bob wishes to send message $M$ to Alice

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

**Encryption** Bob wishes to send message $M$ to Alice

1. He turns $M$ into integer $m$, $0 \leq m < n$, using an agreed-upon reversible protocol known as a padding scheme

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

**Encryption** Bob wishes to send message $M$ to Alice

1. He turns $M$ into integer $m$, $0 \leq m < n$, using an agreed-upon reversible protocol known as a padding scheme
2. He computes the ciphertext $c$ corresponding to $c = m^e \bmod n$. (This can be done quickly)

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

**Encryption** Bob wishes to send message $M$ to Alice

1. He turns $M$ into integer $m$, $0 \leq m < n$, using an agreed-upon reversible protocol known as a padding scheme
2. He computes the ciphertext $c$ corresponding to $c = m^e \bmod n$. (This can be done quickly)
3. Bob transmits $c$ to Alice.

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

**Encryption** Bob wishes to send message $M$ to Alice

1. He turns $M$ into integer $m$, $0 \leq m < n$, using an agreed-upon reversible protocol known as a padding scheme
2. He computes the ciphertext $c$ corresponding to $c = m^e \bmod n$. (This can be done quickly)
3. Bob transmits $c$ to Alice.

**Decryption** Alice can recover $m$ from $c$

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

**Encryption** Bob wishes to send message $M$ to Alice

1. He turns $M$ into integer $m$, $0 \leq m < n$, using an agreed-upon reversible protocol known as a padding scheme
2. He computes the ciphertext $c$ corresponding to $c = m^e \bmod n$. (This can be done quickly)
3. Bob transmits $c$ to Alice.

**Decryption** Alice can recover $m$ from $c$

1. Using her private key exponent $d$ via computing $m = c^d \bmod n$

# RSA: encryption and decryption

Alice transmits her public key $(n, e)$ to Bob and keeps the private key $(n, d)$ secret

**Encryption** Bob wishes to send message $M$ to Alice

1. He turns $M$ into integer $m$, $0 \leq m < n$, using an agreed-upon reversible protocol known as a padding scheme
2. He computes the ciphertext $c$ corresponding to $c = m^e \bmod n$. (This can be done quickly)
3. Bob transmits $c$ to Alice.

**Decryption** Alice can recover $m$ from $c$

1. Using her private key exponent $d$ via computing $m = c^d \bmod n$
2. Given $m$, she can recover the original message $M$ by reversing the padding scheme

# Example

- $n = 43 \cdot 59 = 2537$

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$
- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$
- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$
- Encrypt STOP as two blocks 1819 for ST and 1415 for OP (padding scheme: position in alphabet - 1)

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$
- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$
- Encrypt STOP as two blocks 1819 for ST and 1415 for OP (padding scheme: position in alphabet - 1)
- So, $1819^{13}$ mod 2537 = 2081 and $1415^{13}$ mod 2537 = 2182

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$
- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$
- Encrypt STOP as two blocks 1819 for ST and 1415 for OP (padding scheme: position in alphabet - 1)
- So, $1819^{13}$ mod 2537 = 2081 and $1415^{13}$ mod 2537 = 2182
- So encrypted message is 2081 2182

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$
- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$
- Encrypt STOP as two blocks 1819 for ST and 1415 for OP (padding scheme: position in alphabet - 1)
- So, $1819^{13}$ mod 2537 = 2081 and $1415^{13}$ mod 2537 = 2182
- So encrypted message is 2081 2182
- Receive message 0981 0461: decrypt it

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$
- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$
- Encrypt STOP as two blocks 1819 for ST and 1415 for OP (padding scheme: position in alphabet - 1)
- So, $1819^{13}$ mod 2537 = 2081 and $1415^{13}$ mod 2537 = 2182
- So encrypted message is 2081 2182
- Receive message 0981 0461: decrypt it
- $0981^{937} \bmod 2537 = 0704$ and $0461^{937} \bmod 2537 = 1115$

# Example

- $n = 43 \cdot 59 = 2537$
- $\gcd(13, 42 \cdot 58) = 1$, so public key is $(2537, 13)$
- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$
- Encrypt STOP as two blocks 1819 for ST and 1415 for OP (padding scheme: position in alphabet - 1)
- So, $1819^{13}$ mod 2537 = 2081 and $1415^{13}$ mod 2537 = 2182
- So encrypted message is 2081 2182
- Receive message 0981 0461: decrypt it
- $0981^{937} \bmod 2537 = 0704$ and $0461^{937} \bmod 2537 = 1115$
- So decrypted message is HELP

# RSA: correctness of decryption

Given that $c = m^e$ mod n, is $m = c^d$ mod $n$?

$$c^d = (m^e)^d \equiv m^{ed} \pmod{n}$$

By construction, $d$ and $e$ are each others multiplicative inverses modulo $k$, i.e. $ed \equiv 1 \pmod{k}$. Also $k = (p-1)(q-1)$. Thus $ed - 1 = h(p-1)(q-1)$ for some integer $h$. We consider $m^{ed}$ mod $p$
If $p \not| m$ then
$m^{ed} = m^{h(p-1)(q-1)}m = (m^{p-1})^{h(q-1)}m \equiv 1^{h(q-1)}m \equiv m \pmod{p}$ (by Fermat's little theorem)
Otherwise $m^{ed} \equiv 0 \equiv m \pmod{p}$
Symmetrically, $m^{ed} \equiv m \pmod{q}$
Since $p$, $q$ are distinct primes, we have $m^{ed} \equiv m \pmod{pq}$. Since $n = pq$, we have $c^d = m^{ed} \equiv m \pmod{n}$