Discrete Mathematics & Mathematical Reasoning
Cardinality

Colin Stirling

Informatics

Finite and infinite sets

- $A = \{1, 2, 3\}$ is a finite set with 3 elements

Finite and infinite sets

- $A = \{1, 2, 3\}$ is a finite set with 3 elements
- $B = \{a, b, c, d\}$ and $C = \{1, 2, 3, 4\}$ are finite sets with 4 elements

Finite and infinite sets

- $A = \{1, 2, 3\}$ is a finite set with 3 elements
- $B = \{a, b, c, d\}$ and $C = \{1, 2, 3, 4\}$ are finite sets with 4 elements
- For finite sets, $|X| \leq |Y|$ iff there is an injection $f : X \to Y$

Finite and infinite sets

- $A = \{1, 2, 3\}$ is a finite set with 3 elements
- $B = \{a, b, c, d\}$ and $C = \{1, 2, 3, 4\}$ are finite sets with 4 elements
- For finite sets, $|X| \leq |Y|$ iff there is an injection $f : X \to Y$
- For finite sets, $|X| = |Y|$ iff there is an bijection $f : X \to Y$

Finite and infinite sets

- $A = \{1, 2, 3\}$ is a finite set with 3 elements
- $B = \{a, b, c, d\}$ and $C = \{1, 2, 3, 4\}$ are finite sets with 4 elements
- For finite sets, $|X| \leq |Y|$ iff there is an injection $f : X \to Y$
- For finite sets, $|X| = |Y|$ iff there is an bijection $f : X \to Y$
- $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are infinite sets

Finite and infinite sets

- $A = \{1, 2, 3\}$ is a finite set with 3 elements
- $B = \{a, b, c, d\}$ and $C = \{1, 2, 3, 4\}$ are finite sets with 4 elements
- For finite sets, $|X| \leq |Y|$ iff there is an injection $f : X \to Y$
- For finite sets, $|X| = |Y|$ iff there is an bijection $f : X \to Y$
- $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are infinite sets
- When do two infinite sets have the same size?

Finite and infinite sets

- $A = \{1, 2, 3\}$ is a finite set with 3 elements
- $B = \{a, b, c, d\}$ and $C = \{1, 2, 3, 4\}$ are finite sets with 4 elements
- For finite sets, $|X| \leq |Y|$ iff there is an injection $f : X \to Y$
- For finite sets, $|X| = |Y|$ iff there is an bijection $f : X \to Y$
- $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are infinite sets
- When do two infinite sets have the same size?
- Same answer

Cardinality of sets

Definition

- Two sets A and B have the same cardinality, $|A| = |B|$, iff there exists a bijection from A to B

Cardinality of sets

Definition

- Two sets A and B have the same cardinality, $|A| = |B|$, iff there exists a bijection from A to B
- $|A| \leq |B|$ iff there exists an injection from A to B

Cardinality of sets

Definition

- Two sets A and B have the same cardinality, $|A| = |B|$, iff there exists a bijection from A to B
- $|A| \leq |B|$ iff there exists an injection from A to B
- $|A| < |B|$ iff $|A| \leq |B|$ and $|A| \neq |B|$ (A smaller cardinality than B)

Cardinality of sets

Definition

- Two sets A and B have the same cardinality, $|A| = |B|$, iff there exists a bijection from A to B
- $|A| \leq |B|$ iff there exists an injection from A to B
- $|A| < |B|$ iff $|A| \leq |B|$ and $|A| \neq |B|$ (A smaller cardinality than B)

Unlike finite sets, for infinite sets $A \subset B$ and $|A| = |B|$

Cardinality of sets

Definition

- Two sets A and B have the same cardinality, $|A| = |B|$, iff there exists a bijection from A to B
- $|A| \leq |B|$ iff there exists an injection from A to B
- $|A| < |B|$ iff $|A| \leq |B|$ and $|A| \neq |B|$ (A smaller cardinality than B)

Unlike finite sets, for infinite sets $A \subset B$ and $|A| = |B|$

$\text{Even} = \{2n \mid n \in \mathbb{N}\} \subset \mathbb{N}$ and $|\text{Even}| = |\mathbb{N}|$

Cardinality of sets

Definition

- Two sets A and B have the same cardinality, $|A| = |B|$, iff there exists a bijection from A to B
- $|A| \leq |B|$ iff there exists an injection from A to B
- $|A| < |B|$ iff $|A| \leq |B|$ and $|A| \neq |B|$ (A smaller cardinality than B)

Unlike finite sets, for infinite sets $A \subset B$ and $|A| = |B|$

$\text{Even} = \{2n \mid n \in \mathbb{N}\} \subset \mathbb{N}$ and $|\text{Even}| = |\mathbb{N}|$

$f : \mathit{Even} \rightarrow \mathbb{N}$ with $f(2n) = n$ is a bijection

Countable sets

Definition

- A set S is called countably infinite, iff it has the same cardinality as the positive integers, $|\mathbb{Z}^+| = |S|$

Countable sets

Definition

- A set S is called countably infinite, iff it has the same cardinality as the positive integers, $|\mathbb{Z}^+| = |S|$
- A set is called countable iff it is either finite or countably infinite

Countable sets

Definition

- A set S is called countably infinite, iff it has the same cardinality as the positive integers, $|\mathbb{Z}^+| = |S|$
- A set is called countable iff it is either finite or countably infinite
- A set that is not countable is called uncountable

Countable sets

Definition

- A set S is called countably infinite, iff it has the same cardinality as the positive integers, $|\mathbb{Z}^+| = |S|$
- A set is called countable iff it is either finite or countably infinite
- A set that is not countable is called uncountable

$\mathbb{N}$ is countably infinite; what is the bijection $f : \mathbb{Z}^+ \rightarrow \mathbb{N}$?

Countable sets

Definition

- A set S is called countably infinite, iff it has the same cardinality as the positive integers, $|\mathbb{Z}^+| = |S|$
- A set is called countable iff it is either finite or countably infinite
- A set that is not countable is called uncountable

$\mathbb{N}$ is countably infinite; what is the bijection $f : \mathbb{Z}^+ \to \mathbb{N}$?

$\mathbb{Z}$ is countably infinite; what is the bijection $g : \mathbb{Z}^+ \to \mathbb{Z}$?

The positive rational numbers are countable

Construct a bijection $f : \mathbb{Z}^+ \rightarrow \mathbb{Q}^+$

The positive rational numbers are countable

Construct a bijection $f : \mathbb{Z}^+ \rightarrow \mathbb{Q}^+$

List fractions p/q with $q = n$ in the n^{th} row

The positive rational numbers are countable

Construct a bijection $f : \mathbb{Z}^+ \to \mathbb{Q}^+$

List fractions p/q with $q = n$ in the n^{th} row

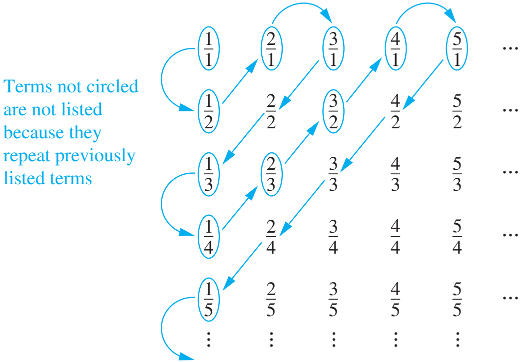
f traverses this list in the order for $m = 2, 3, 4, \ldots$ visiting all p/q with $p + q = m$ (and listing only new rationals)

The positive rational numbers are countable

Construct a bijection $f : \mathbb{Z}^+ \to \mathbb{Q}^+$

List fractions p/q with $q = n$ in the n^{th} row

f traverses this list in the order for $m = 2, 3, 4, \ldots$ visiting all p/q with $p + q = m$ (and listing only new rationals)



Countable sets

Theorem

If A and B are countable sets, then $A \cup B$ is countable

Countable sets

Theorem

If A and B are countable sets, then $A \cup B$ is countable

Proof in book

Countable sets

Theorem

If A and B are countable sets, then $A \cup B$ is countable

Proof in book

Theorem

If I is countable and for each $i \in I$ the set A_i is countable then $\bigcup_{i \in I} A_i$ is countable

Countable sets

Theorem

If A and B are countable sets, then $A \cup B$ is countable

Proof in book

Theorem

If I is countable and for each $i \in I$ the set A_i is countable then $\bigcup_{i \in I} A_i$ is countable

Proof in book

Finite strings

Theorem

The set Σ^* of all finite strings over a finite alphabet Σ is countably infinite

Finite strings

Theorem

The set Σ^* of all finite strings over a finite alphabet Σ is countably infinite

Proof.

- First define an (alphabetical) ordering on the symbols in Σ Show that the strings can be listed in a sequence
  - First single string ε of length 0
  - Then all strings of length 1 in lexicographic order
  - Then all strings of length 2 in lexicographic order
  - $\vdots$
  - $\vdots$

Finite strings

Theorem

The set Σ^* of all finite strings over a finite alphabet Σ is countably infinite

Proof.

- First define an (alphabetical) ordering on the symbols in Σ
  Show that the strings can be listed in a sequence
  - First single string ε of length 0
  - Then all strings of length 1 in lexicographic order
  - Then all strings of length 2 in lexicographic order
  - $\vdots$
  - $\vdots$
- Each of these sets is countable; so is their union Σ^*

$\square$

Finite strings

Theorem

The set Σ^* of all finite strings over a finite alphabet Σ is countably infinite

Proof.

- First define an (alphabetical) ordering on the symbols in Σ
  Show that the strings can be listed in a sequence
  - First single string ε of length 0
  - Then all strings of length 1 in lexicographic order
  - Then all strings of length 2 in lexicographic order
  - $\vdots$
  - $\vdots$
- Each of these sets is countable; so is their union Σ^*

$\square$

The set of Java-programs is countable; a program is just a finite string

Infinite binary strings

- An infinite length string of bits 10010 . . .

Infinite binary strings

- An infinite length string of bits 10010 . . .
- Such a string is a function $d : \mathbb{Z}^+ \rightarrow \{0, 1\}$

Infinite binary strings

- An infinite length string of bits 10010 . . .
- Such a string is a function $d : \mathbb{Z}^+ \to \{0, 1\}$
- With the property $d_m = d(m)$ is the mth symbol

Uncountable sets

Theorem

The set of infinite binary strings is uncountable

Uncountable sets

Theorem

The set of infinite binary strings is uncountable

Proof.

Let X be the set of infinite binary strings. For a contradiction assume that a bijection $f : \mathbb{Z}^+ \to X$ exists. So, f must be onto (surjective). Assume that $f(i) = d^i$ for $i \in \mathbb{Z}^+$. So, $X = \{d^1, d^2, \ldots, d^m, \ldots\}$. Define the infinite binary string d as follows: $d_n = (d^n_n + 1)$ mod 2. But for each m, $d \neq d^m$ because $d_m \neq d^m_m$. So, f is not a surjection. $\square$

Uncountable sets

Theorem

The set of infinite binary strings is uncountable

Proof.

Let X be the set of infinite binary strings. For a contradiction assume that a bijection $f : \mathbb{Z}^+ \to X$ exists. So, f must be onto (surjective). Assume that $f(i) = d^i$ for $i \in \mathbb{Z}^+$. So, $X = \{d^1, d^2, \ldots, d^m, \ldots\}$. Define the infinite binary string d as follows: $d_n = (d^n_n + 1) \bmod 2$. But for each m, $d \neq d^m$ because $d_m \neq d^m_m$. So, f is not a surjection. □

The technique used here is called diagonalization

Uncountable sets

Theorem

The set of infinite binary strings is uncountable

Proof.

Let X be the set of infinite binary strings. For a contradiction assume that a bijection $f : \mathbb{Z}^+ \to X$ exists. So, f must be onto (surjective). Assume that $f(i) = d^i$ for $i \in \mathbb{Z}^+$. So, $X = \{d^1, d^2, \ldots, d^m, \ldots\}$. Define the infinite binary string d as follows: $d_n = (d^n_n + 1) \bmod 2$. But for each m, $d \neq d^m$ because $d_m \neq d^m_m$. So, f is not a surjection. $\square$

The technique used here is called diagonalization

Uncountable sets

Theorem

The set of infinite binary strings is uncountable

Proof.

Let X be the set of infinite binary strings. For a contradiction assume that a bijection $f : \mathbb{Z}^+ \to X$ exists. So, f must be onto (surjective). Assume that $f(i) = d^i$ for $i \in \mathbb{Z}^+$. So, $X = \{d^1, d^2, \ldots, d^m, \ldots\}$. Define the infinite binary string d as follows: $d_n = (d^n_n + 1) \bmod 2$. But for each m, $d \neq d^m$ because $d_m \neq d^m_m$. So, f is not a surjection. $\square$

The technique used here is called diagonalization

Similar argument shows that $\mathbb{R}$ via $[0, 1]$ is uncountable using infinite decimal strings (see book)

More on the uncountable

Corollary

The set of functions $F = \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$ is uncountable

More on the uncountable

Corollary

The set of functions $F = \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$ is uncountable

The set of functions $C = \{f \mid f : \mathbb{Z} \to \mathbb{Z} \text{ is computable}\}$ is countable

More on the uncountable

Corollary

The set of functions $F = \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$ is uncountable

The set of functions $C = \{f \mid f : \mathbb{Z} \to \mathbb{Z} \text{ is computable}\}$ is countable

Therefore, “most functions” in F are not computable!

Schröder-Bernstein Theorem

Theorem

If $|A| \leq |B|$ *and* $|B| \leq |A|$ *then* $|A| = |B|$

Schröder-Bernstein Theorem

Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

- Example $|(0,1)| = |(0,1]|$

Schröder-Bernstein Theorem

Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

- Example $|(0,1)| = |(0,1]|$
- $|(0,1)| \leq |(0,1]|$ using identity function

Schröder-Bernstein Theorem

Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

- Example $|(0,1)| = |(0,1]|$
- $|(0,1)| \leq |(0,1]|$ using identity function
- $|(0,1]| \leq |(0,1)|$ use $f(x) = x/2$ as $(0, 1/2] \subset (0,1)$

Cantor's theorem

Theorem

$|A| < |\mathcal{P}(A)|$

Cantor's theorem

Theorem

$|A| < |\mathcal{P}(A)|$

Proof.

Consider the injection $f : A \to \mathcal{P}(A)$ with $f(a) = \{a\}$ for any $a \in A$. Therefore, $|A| \leq |\mathcal{P}(A)|$. Next we show there is not a surjection $f : A \to \mathcal{P}(A)$. For a contradiction, assume that a surjection f exists. We define the set $B \subseteq A$: $B = \{x \in A \mid x \notin f(x)\}$. Since f is a surjection, there must exist an $a \in A$ s.t. $B = f(a)$. Now there are two cases:

1. If $a \in B$ then, by definition of B, $a \notin B = f(a)$. Contradiction
2. If $a \notin B$ then $a \notin f(a)$; by definition of B, $a \in B$. Contradiction

□

Implications of Cantor's theorem

- $\mathcal{P}(\mathbb{N})$ is not countable (in fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)

Implications of Cantor's theorem

- $\mathcal{P}(\mathbb{N})$ is not countable (in fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)
- The Continuum Hypothesis claims there is no set S with $|\mathbb{N}| < |S| < |\mathbb{R}|$

Implications of Cantor's theorem

- $\mathcal{P}(\mathbb{N})$ is not countable (in fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)
- The Continuum Hypothesis claims there is no set S with $|\mathbb{N}| < |S| < |\mathbb{R}|$
- It was 1st of Hilbert's 23 open problems presented in 1900. Shown to be independent of ZF set theory by Gödel/Cohen in 1963: cannot be proven/disproven in ZF

Implications of Cantor's theorem

- $\mathcal{P}(\mathbb{N})$ is not countable (in fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)
- The Continuum Hypothesis claims there is no set S with $|\mathbb{N}| < |S| < |\mathbb{R}|$
- It was 1st of Hilbert's 23 open problems presented in 1900. Shown to be independent of ZF set theory by Gödel/Cohen in 1963: cannot be proven/disproven in ZF
- There exists an infinite hierarchy of sets of ever larger cardinality

Implications of Cantor's theorem

- $\mathcal{P}(\mathbb{N})$ is not countable (in fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)
- The Continuum Hypothesis claims there is no set S with $|\mathbb{N}| < |S| < |\mathbb{R}|$
- It was 1st of Hilbert's 23 open problems presented in 1900. Shown to be independent of ZF set theory by Gödel/Cohen in 1963: cannot be proven/disproven in ZF
- There exists an infinite hierarchy of sets of ever larger cardinality
- $S_0 = \mathbb{N}$ and $S_{i+1} = \mathcal{P}(S_i)$

Implications of Cantor's theorem

- $\mathcal{P}(\mathbb{N})$ is not countable (in fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)
- The Continuum Hypothesis claims there is no set S with $|\mathbb{N}| < |S| < |\mathbb{R}|$
- It was 1st of Hilbert's 23 open problems presented in 1900. Shown to be independent of ZF set theory by Gödel/Cohen in 1963: cannot be proven/disproven in ZF
- There exists an infinite hierarchy of sets of ever larger cardinality
- $S_0 = \mathbb{N}$ and $S_{i+1} = \mathcal{P}(S_i)$
- $|S_0| < |S_1| < \ldots < |S_i| < |S_{i+1}| < \ldots$