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Definition

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$f : \text{Even} \rightarrow \mathbb{N}$ with $f(2n) = n$ is a bijection
A set $S$ is called countably infinite, iff it has the same cardinality as the positive integers, $|\mathbb{Z}^+| = |S|$.
Countable sets

Definition

- A set $S$ is called countably infinite, iff it has the same cardinality as the positive integers, $|\mathbb{Z}^+| = |S|$
- A set is called countable iff it is either finite or countably infinite

$\mathbb{N}$ is countably infinite; what is the bijection $f: \mathbb{Z}^+ \rightarrow \mathbb{N}$?

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$f$ traverses this list in the order for $m = 2, 3, 4, \ldots$ visiting all $p/q$ with $p + q = m$ (and listing only new rationals)
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Terms not circled are not listed because they repeat previously listed terms
Theorem

If $A$ and $B$ are countable sets, then $A \cup B$ is countable
Countable sets

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Proof in book
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If $I$ is countable and for each $i \in I$ the set $A_i$ is countable then $\bigcup_{i \in I} A_i$ is countable
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Finite strings

Theorem

The set $\Sigma^*$ of all finite strings over a finite alphabet $\Sigma$ is countably infinite

Proof.

First define an (alphabetical) ordering on the symbols in $\Sigma$

Show that the strings can be listed in a sequence

▶ First single string $\varepsilon$ of length 0

▶ Then all strings of length 1 in lexicographic order

▶ Then all strings of length 2 in lexicographic order

▶ ...

Each of these sets is countable; so is their union $\Sigma^*$

The set of Java-programs is countable; a program is just a finite string
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- With the property \( d_m = d(m) \) is the \( m \)th symbol
Uncountable sets

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*The set of infinite binary strings is uncountable*

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Let $X$ be the set of infinite binary strings. For a contradiction assume that a bijection $f: \mathbb{Z}^+ \rightarrow X$ exists. So, $f$ must be onto (surjective).

Assume that $f(i) = d_i$ for $i \in \mathbb{Z}^+$. So, $X = \{d_1, d_2, \ldots, d_m, \ldots\}$. Define the infinite binary string $d$ as follows: $d_n = (d_{n+1}) \mod 2$. But for each $m$, $d \neq d_m$ because $d_m \neq d_m$. So, $f$ is not a surjection.

The technique used here is called diagonalization. Similar argument shows that $\mathbb{R}$ via $[0,1]$ is uncountable using infinite decimal strings (see book).
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Similar argument shows that $\mathbb{R}$ via $[0, 1]$ is uncountable using infinite decimal strings (see book)
The set of functions $F = \{ f \mid f : \mathbb{Z} \rightarrow \mathbb{Z} \}$ is uncountable.
More on the uncountable

Corollary

The set of functions $F = \{ f \mid f : \mathbb{Z} \to \mathbb{Z} \}$ is uncountable

The set of functions $C = \{ f \mid f : \mathbb{Z} \to \mathbb{Z} \text{ is computable} \}$ is countable
Corollary

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Therefore, “most functions” in \( F \) are not computable!
Schröder-Bernstein Theorem

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\[ |A| \leq |B| \text{ and } |B| \leq |A| \text{ then } |A| = |B| \]
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- **Example** $|(0, 1)| = |(0, 1]|$
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- **Example** $|(0, 1)| = |(0, 1]|$
- $|(0, 1)| \leq |(0, 1]|$ using identity function
- $|(0, 1]| \leq |(0, 1)|$ use $f(x) = x/2$ as $(0, 1/2] \subset (0, 1)$
Cantor’s theorem

Theorem

\[ |A| < |\mathcal{P}(A)| \]
**Cantor’s theorem**

**Theorem**

\[ |A| < |\mathcal{P}(A)| \]

**Proof.**

Consider the injection \( f : A \to \mathcal{P}(A) \) with \( f(a) = \{a\} \) for any \( a \in A \). Therefore, \( |A| \leq |\mathcal{P}(A)| \). Next we show there is not a surjection \( f : A \to \mathcal{P}(A) \). For a contradiction, assume that a surjection \( f \) exists. We define the set \( B \subseteq A : B = \{ x \in A \mid x \notin f(x) \} \). Since \( f \) is a surjection, there must exist an \( a \in A \) s.t. \( B = f(a) \). Now there are two cases:

1. If \( a \in B \) then, by definition of \( B \), \( a \notin B = f(a) \). Contradiction
2. If \( a \notin B \) then \( a \notin f(a) \); by definition of \( B \), \( a \in B \). Contradiction
Implications of Cantor’s theorem

- $\mathcal{P}(\mathbb{N})$ is not countable (in fact, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)

The Continuum Hypothesis claims there is no set $S$ with $|\mathbb{N}| < |S| < |\mathbb{R}|$.

It was 1st of Hilbert's 23 open problems presented in 1900. Shown to be independent of ZF set theory by Gödel/Cohen in 1963: cannot be proven/disproven in ZF.

There exists an infinite hierarchy of sets of ever larger cardinality $S_0 = \mathbb{N}$ and $S_{i+1} = \mathcal{P}(S_i)$:

$|S_0| < |S_1| < \ldots < |S_i| < |S_{i+1}| < \ldots$
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