Discrete Mathematics & Mathematical Reasoning Basic Structures: Sets, Functions, Relations, Sequences and Sums

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Informatics

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- Closed intervals $[0,1] = \{r \mid 0 \leq r \leq 1\}$

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- $A \times B$ cartesian product (tuple sets)

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- Modern formulations (such as Zermelo-Fraenkel set theory) restrict comprehension. (However, it is impossible to prove in ZF that ZF is consistent unless ZF is inconsistent.)

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- $f : A \rightarrow B$ if f is a function from A to B

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For $f : A \rightarrow B$, A is the domain and B is codomain

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One-to-one correspondence or bijection

Definition

$f : A \rightarrow B$ is a bijection iff it is both injective and surjective

- Is the identity function $\iota_A : A \rightarrow A$ a bijection? YES
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Function composition

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Let $f : B \to C$ and $g : A \to B$. The composition function $f \circ g : A \to C$ is $(f \circ g)(a) = f(g(a))$

Results about function composition

Theorem

The composition of two functions is a function

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Corollary

The composition of two bijections is a bijection

Inverse function

Definition

If $f : A \rightarrow B$ is a bijection, then the inverse of f, written $f^{-1} : B \rightarrow A$ is $f^{-1}(b) = a$ iff $f(a) = b$

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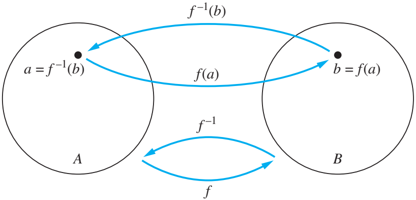

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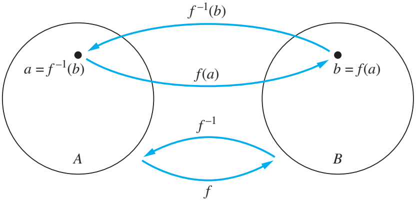

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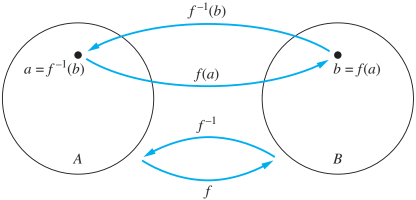

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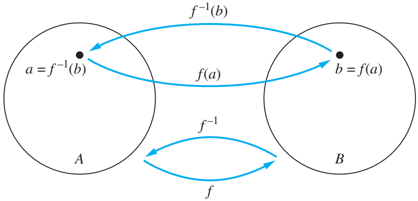

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What is $f^{-1} \circ f$? and $f \circ f^{-1}$?

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Definition

Given sets $A_1, \ldots, A_n$ a subset $R \subseteq A_1 \times \cdots \times A_n$ is an n-ary relation

Examples

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- $R \subseteq S$ subset and $R = S$ equality

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Let $R \subseteq B \times C$ and $S \subseteq A \times B$. The composition relation $(R \circ S) \subseteq A \times C$ is $\{(a, c) \mid \exists b\ (a, b) \in S \wedge (b, c) \in R\}$

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Example: reachability in a graph

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R^* is the reflexive and transitive closure of R

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- For integer $m > 1$ the relation $\equiv \pmod{m}$ is an equivalence relation on integers

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If $b \in [a]_R$ then b is called a representative of the equivalence class

Theorem

Result

Let R be an equivalence relation on A and $a, b \in A$. The following three statements are equivalent

1. aRb
2. $[a]_R = [b]_R$
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Proof in book

Partitions of a set

Definition

A partition of a set A is a collection of disjoint, nonempty subsets that have A as their union. In other words, the collection of subsets $A_i \subseteq A$ with $i \in I$ (where I is an index set) forms a partition of A iff

1. $A_i \neq \emptyset$ for all $i \in I$
2. $A_i \cap A_j = \emptyset$ for all $i \neq j \in I$
3. $\bigcup_{i \in I} A_i = A$

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1. If R is an equivalence on A, then the equivalence classes of R form a partition of A
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2, 3, 5, 7, 11, 13, 17, 19, ... or $a, b, c, d, \ldots, y, z$

Definition

A sequence over a set S is a function f from a subset of the integers (typically $\mathbb{N}$ or $\mathbb{Z}^+$) to the set S. If the domain of f is finite then the sequence is finite

Examples

$f : \mathbb{Z}^+ \to \mathbb{Q}$ is $f(n) = 1/n$ defines the sequence

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where the initial elements a, the common ratio r and the common difference d are real numbers

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- A sequence is called a solution of a recurrence relation iff its terms satisfy the recurrence relation

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Yields the sequence 1, 1, 2, 3, 5, 8, 13, ...

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- The guess can be proved correct by the method of induction (to be covered)

Iterative solution - working upwards

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\begin{aligned}
a_2 &= 2 + 3 \\
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a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \\
&\vdots \\
a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3 \cdot (n-1)
\end{aligned}$$

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$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

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- $P_{20} = (1.03)^{20}\, 1000 = 1,806$

Common sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Summations

Given a sequence $\{a_n\}$, the sum of terms a_m, a_{m+1}, $\ldots$, a_ℓ is

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The variable j is called the index of summation

More generally for an index set S

$$\sum_{j \in S} a_j$$

Useful summation formulas

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Products

Given a sequence $\{a_n\}$, the product of terms $a_m, a_{m+1}, \ldots, a_\ell$ is

$$a_m \cdot a_{m+1} \cdot \ldots \cdot a_\ell$$

$$\prod_{j=m}^{\ell} a_j \quad \text{or} \quad \prod_{m \leq j \leq \ell} a_j$$

More generally for a finite index set S one writes

$$\prod_{j \in S} a_j$$