Discrete Mathematics & Mathematical Reasoning
Greatest Common Divisors

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Informatics

Greatest common divisor

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Although 9 and 22 are coprime they are both composite

Gcd by prime factorisations

Suppose that the prime factorisations of a and b are

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \qquad\qquad b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$$

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Factorisation is a very inefficient method to compute gcd

Euclidian algorithm: efficient for computing gcd

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    if y = 0
    then return(x)
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The Euclidian algorithm relies on

$$\forall x, y \in \mathbb{Z}^+ \, (\text{gcd}(x, y) = \text{gcd}(y, x \bmod y))$$

Euclidian algorithm (proof of correctness)

Lemma

If $x = yq + r$, where x, y, q, and r are positive integers, then $\gcd(x, y) = \gcd(y, r)$. (Consider $r = x \bmod y$ and $q = x \operatorname{div} y$)

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Proof.

($\Rightarrow$) Suppose that d divides both x and y. Then d also divides $x - yq = r$. Hence, any common divisor of x and y must also be a common divisor of y and r.
($\Leftarrow$) Suppose that d divides both y and r. Then d also divides $yq + r = x$. Hence, any common divisor of y and r must also be a common divisor of x and y.
Therefore, $\gcd(x, y) = \gcd(y, r)$ □

Gcd as a linear combination

Theorem (Bézout's theorem)

If x and y are positive integers, then there exist integers a and b such that $gcd(x, y) = ax + by$

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Proof.

Nonconstructive proof. Let S be the set of positive integers $ax + by$ (where a or b may be negative integers); S is non-empty as it includes $x + y$. By the well-ordering principle S has a least element c. So $c = ax + by$ for some a and b. If $d|x$ and $d|y$ then $d|ax$ and $d|by$ and so $d|(ax + by)$, that is $d|c$. We now show $c|x$ and $c|y$ which means that $c = \gcd(x, y)$. Assume $c \not| x$. So $x = qc + r$ where $0 < r < c$. Now $r = x - qc = x - q(ax + by)$. That is, $r = (1 - qa)x + (-qb)y$, so $r \in S$ which contradicts that c is the least element in S as $r < c$. The same argument shows $c|y$. $\square$

Bézout's theorem: constructive proof

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Extended Euclidian algorithm

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algorithm e-gcd(x,y)
    if y = 0
    then return(x, 1, 0)
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     (d, a, b) := e-gcd(y, x mod y)
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Correctness proof for computing Bézout coefficients

- Let $x = yq + r$ where $r = x \bmod y$ and $q = x \operatorname{div} y$
- So $r = x - yq$

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- If $d = ay + br$ then
  $d = ay + b(x - yq) = bx + (a - qb)y$
- Base case $y = 0$: e-gcd$(x, y) = (x, 1, 0)$ and $x = 1 * x + 0 * y$

Further properties

Theorem

If a, b, c are positive integers such that $\gcd(a, b) = 1$ and $a | bc$ then $a | c$

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If a, b, c are positive integers such that gcd$(a, b) = 1$ and $a|bc$ then $a|c$

Proof.

Because gcd$(a, b) = 1$, by Bézout's theorem there are integers s and t such that $sa + tb = 1$. So, $sac + tbc = c$. Assume $a|bc$. Therefore, $a|tbc$ and $a|sac$, so $a|(sac + tbc)$; that is, $a|c$. $\square$

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Let m be a positive integer and let a, b, c be integers. If $ac \equiv bc \pmod{m}$ and $gcd(c, m) = 1$ then $a \equiv b \pmod{m}$

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Theorem

Let m be a positive integer and let a, b, c be integers. If $ac \equiv bc \pmod m$ and $gcd(c, m) = 1$ then $a \equiv b \pmod m$

Proof.

Because $ac \equiv bc \,(\text{mod } m)$, it follows $m|(ac - bc)$; so, $m|c(a - b)$. By the result above because $\gcd(c, m) = 1$, it follows that $m|(a - b)$. $\square$