Discrete Mathematics & Mathematical Reasoning Basic Structures: Sets, Functions, Relations, Sequences and Sums

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Informatics

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- Subsets of sets upon which an order is defined

$$\begin{array}{lcll}
[a, b] & = & \{x \mid a \leq x \leq b\} & \text{closed interval} \\
[a, b) & = & \{x \mid a \leq x < b\} & \\
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- $A \times B$ cartesian product (tuple sets)

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- Modern formulations (such as Zermelo-Fraenkel set theory) restrict comprehension. (However, it is impossible to prove in ZF that ZF is consistent unless ZF is inconsistent.)

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- $f : A \rightarrow B$ if f is a function from A to B

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Function composition

Definition

Let $f : B \rightarrow C$ and $g : A \rightarrow B$. The composition function $f \circ g : A \rightarrow C$ is $(f \circ g)(a) = f(g(a))$

Results about function composition

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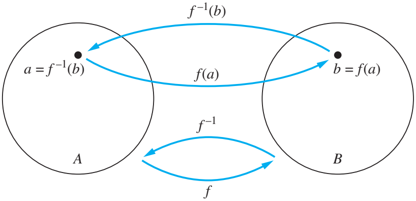
Corollary

The composition of two bijections is a bijection

Inverse function

Definition

If $f : A \rightarrow B$ is a bijection, then the inverse of f, written $f^{-1} : B \rightarrow A$ is $f^{-1}(b) = a$ iff $f(a) = b$



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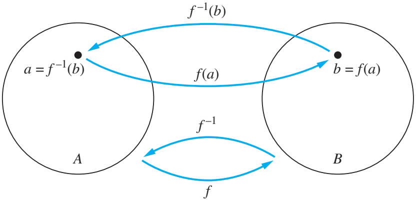

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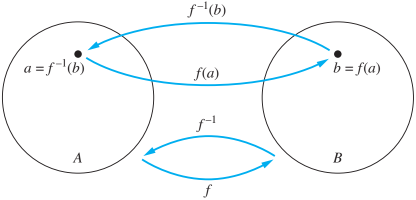

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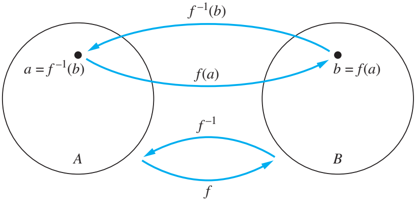

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What is $f^{-1} \circ f$? and $f \circ f^{-1}$?

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$$\lfloor -6.1 \rfloor = -7 \qquad \lceil 6.1 \rceil = 7$$

The factorial function

Definition

The factorial function $f : \mathbb{N} \rightarrow \mathbb{N}$, denoted as $f(n) = n!$ assigns to n the product of the first n positive integers

$$f(0) = 0! = 1$$

and

$$f(n) = n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$$

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- R is a set of tuples (a, b) with $a \in A$ and $b \in B$
- Often we write $a\,R\,b$ for $(a, b) \in R$
- A function f is a restricted relation where

$$\forall a \in A\, \exists b \in B\, (((a, b) \in f) \wedge \forall c \in B\, ((a, c) \in f \rightarrow c = b))$$

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- R is a set of tuples (a, b) with $a \in A$ and $b \in B$
- Often we write $a\,R\,b$ for $(a, b) \in R$
- A function f is a restricted relation where

$$\forall a \in A\, \exists b \in B\, (((a, b) \in f) \wedge \forall c \in B\, ((a, c) \in f \rightarrow c = b))$$

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Definition

Given sets $A_1, \ldots, A_n$ a subset $R \subseteq A_1 \times \cdots \times A_n$ is an n-ary relation

Examples

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- Written as $a \equiv b \pmod{m}$

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- $R \subseteq S$ subset and $R = S$ equality

Relation composition

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Let $R \subseteq B \times C$ and $S \subseteq A \times B$. The composition relation $(R \circ S) \subseteq A \times C$ is $\{(a, c) \mid \exists b \, (a, b) \in S \wedge (b, c) \in R\}$

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Example: reachability in a graph

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- $|$ on integers is not an equivalence relation.
- For integer $m > 1$ the relation $\equiv \pmod{m}$ is an equivalence relation on integers

Equivalence classes

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Let R be an equivalence relation on a set A and $a \in A$. Let

$$[a]_R = \{s \mid (a, s) \in R\}$$

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If $b \in [a]_R$ then b is called a representative of the equivalence class

Theorem

Result

Let R be an equivalence relation on A and $a, b \in A$. The following three statements are equivalent

1. aRb
2. $[a]_R = [b]_R$
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Proof in book

Partitions of a set

Definition

A partition of a set A is a collection of disjoint, nonempty subsets that have A as their union. In other words, the collection of subsets $A_i \subseteq A$ with $i \in I$ (where I is an index set) forms a partition of A iff

1. $A_i \neq \emptyset$ for all $i \in I$
2. $A_i \cap A_j = \emptyset$ for all $i \neq j \in I$
3. $\bigcup_{i \in I} A_i = A$

Result

Theorem

1. If R is an equivalence on A, then the equivalence classes of R form a partition of A
2. Conversely, given a partition $\{A_i \mid i \in I\}$ of A there exists an equivalence relation R that has exactly the sets $A_i, i \in I$, as its equivalence classes

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Definition

A sequence over a set S is a function f from a subset of the integers (typically $\mathbb{N}$ or $\mathbb{Z}^+$) to the set S. If the domain of f is finite then the sequence is finite

Examples

$f : \mathbb{Z}^+ \to \mathbb{Q}$ is $f(n) = 1/n$ defines the sequence

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Geometric and arithmetic progressions

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where the initial elements a, the common ratio r and the common difference d are real numbers

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- A sequence is called a solution of a recurrence relation iff its terms satisfy the recurrence relation

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Yields the sequence 1, 1, 2, 3, 5, 8, 13, ...

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- The guess can be proved correct by the method of induction (to be covered)

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a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3 \cdot (n - 1)
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- $P_1 = (1.03)\, P_0, \ldots, \quad P_n = (1.03)P_{n-1} \quad = (1.03)^n P_0$
- $P_{20} = (1.03)^{20}\, 1000 = 1,806$

Common sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Summations

Given a sequence $\{a_n\}$, the sum of terms a_m, a_{m+1}, $\ldots$, a_ℓ is

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The variable j is called the index of summation

More generally for an index set S

$$\sum_{j \in S} a_j$$

Useful summation formulas

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \|x\| < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \|x\| < 1$	$\frac{1}{(1-x)^2}$

Products

Given a sequence $\{a_n\}$, the product of terms $a_m, a_{m+1}, \ldots, a_\ell$ is

$$a_m \cdot a_{m+1} \cdot \ldots \cdot a_\ell$$

$$\prod_{j=m}^{\ell} a_j \quad \text{or} \quad \prod_{m \leq j \leq \ell} a_j$$

More generally for a finite index set S one writes

$$\prod_{j \in S} a_j$$