Discrete Mathematics & Mathematical Reasoning
Predicates, Quantifiers and Proof Techniques

Colin Stirling
Informatics
Recall propositional logic from last year (in Inf1CL)

Propositions can be constructed from other propositions using logical connectives

- Negation: \( \neg \)
- Conjunction: \( \land \)
- Disjunction: \( \lor \)
- Implication: \( \rightarrow \)
- Biconditional: \( \leftrightarrow \)

The truth of a proposition is defined by the truth values of its elementary propositions and the meaning of connectives.

The meaning of logical connectives can be defined using truth tables.
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- Disjunction: $\lor$
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Propositions can be constructed from other propositions using logical connectives

- Negation: ¬
- Conjunction: ∧
- Disjunction: ∨
- Implication: →
- Biconditional: ↔

The truth of a proposition is defined by the truth values of its elementary propositions and the meaning of connectives
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- Disjunction: \( \lor \)
- Implication: \( \rightarrow \)
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The truth of a proposition is defined by the truth values of its elementary propositions and the meaning of connectives

The meaning of logical connectives can be defined using truth tables
Examples of logical implication and equivalence

- \((p \land (p \rightarrow q)) \rightarrow q\)
- \((p \land \neg p) \rightarrow q\)
- \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)
- \(\vdash\)
Examples of logical implication and equivalence

- \((p \land (p \rightarrow q)) \rightarrow q\)
- \((p \land \neg p) \rightarrow q\)
- \(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)\)

: 
- \((p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)\)
- \((p \land q) \leftrightarrow (\neg p \lor \neg q)\)  \(\text{De Morgan}\)
- \((p \lor q) \leftrightarrow (\neg p \land \neg q)\)  \(\text{De Morgan}\)
- \((p \rightarrow q) \leftrightarrow (p \land \neg q)\)

: 

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Discrete Mathematics (Chap 1)
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Propositional logic is not “enough”
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In propositional logic, from

- All humans are mortal \((\text{proposition } p)\)
- Socrates is human \((\text{proposition } q)\)

we cannot derive

- Socrates is mortal \((\text{proposition } r)\)
Propositional logic is not “enough”

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- All humans are mortal \((\text{proposition } p)\)
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- Socrates is mortal \((\text{proposition } r)\)

\((p \land q) \rightarrow r\) is not a tautology
Propositional logic is not “enough”

In propositional logic, from

- All humans are mortal  (proposition $p$)
- Socrates is human    (proposition $q$)

we cannot derive

- Socrates is mortal  (proposition $r$)

- $(p \land q) \rightarrow r$ is not a tautology

We need a language to talk about objects, their properties and their relations
Predicate logic

Extends propositional logic by the new features

- Variables: $x, y, z, \ldots$
- Predicates: $P(x), Q(x), R(x, y), M(x, y, z), \ldots$
- Quantifiers: $\forall, \exists$

Predicates are a generalisation of propositions

Can contain variables

Variables stand for (and can be replaced by) elements from their domain

The truth value of a predicate depends on the values of its variables
Predicate logic

Extends propositional logic by the new features

- Variables: \( x, y, z, \ldots \)
- Predicates: \( P(x), Q(x), R(x, y), M(x, y, z), \ldots \)
- Quantifiers: \( \forall, \exists \)

Predicates are a generalisation of propositions

- Can contain variables \( M(x, y, z) \)
- Variables stand for (and can be replaced by) elements from their domain
- The truth value of a predicate depends on the values of its variables
$P(x)$ is “$x > 5$” and $x$ ranges over $\mathbb{Z}$ (integers)

- $P(8)$ is true
- $P(-1)$ is false
Examples

$P(x)$ is “$x > 5$” and $x$ ranges over $\mathbb{Z}$ (integers)

- $P(8)$ is true
- $P(-1)$ is false

$H(x)$ is “$x$ is human”; $M(x)$ is “$x$ is mortal” and $x$ ranges over animals

- $M(Socrates)$ is true
- $H(Sansa)$ is false
Examples

$P(x)$ is “$x > 5$” and $x$ ranges over $\mathbb{Z}$ (integers)
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$H(x)$ is “$x$ is human”; $M(x)$ is “$x$ is mortal” and $x$ ranges over animals
- $M(Socrates)$ is true
- $H(Sansa)$ is false

$D(x, y)$ is “$x$ divides $y$” and $x, y$ range over $\mathbb{Z}^+$ (positive integers)
- $D(3, 9)$ is true
- $D(2, 9)$ is false
Examples

$P(x)$ is “$x > 5$” and $x$ ranges over $\mathbb{Z}$ (integers)
- $P(8)$ is true
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$D(x, y)$ is “$x$ divides $y$” and $x, y$ range over $\mathbb{Z}^+$ (positive integers)
- $D(3, 9)$ is true
- $D(2, 9)$ is false

$S(x_1, \ldots, x_{11}, y)$ is “$x_1 + \ldots + x_{11}$ is $y$”
Quantifiers

- Universal quantifier, “For all”: ∀
  ∀x P(x) asserts that P(x) is true for every x in the assumed domain
Quantifiers

- **Universal quantifier**, “For all”: $\forall$
  $\forall x \; P(x)$ asserts that $P(x)$ is true for every $x$ in the assumed domain

- **Existential quantifier**, “There exists”: $\exists$
  $\exists x \; P(x)$ asserts that $P(x)$ is true for some $x$ in the assumed domain
Quantifiers

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  $\forall x\ P(x)$ asserts that $P(x)$ is true for every $x$ in the assumed domain

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  $\exists x\ P(x)$ asserts that $P(x)$ is true for some $x$ in the assumed domain

The quantifiers are said to bind the variable $x$ in these expressions. Variables in the scope of some quantifier are called bound variables. All other variables in the expression are called free variables.
Quantifiers

- **Universal quantifier, “For all”:** \( \forall \)
  - \( \forall x \ P(x) \) asserts that \( P(x) \) is true for every \( x \) in the assumed domain

- **Existential quantifier, “There exists”:** \( \exists \)
  - \( \exists x \ P(x) \) asserts that \( P(x) \) is true for some \( x \) in the assumed domain

- The quantifiers are said to bind the variable \( x \) in these expressions. Variables in the scope of some quantifier are called bound variables. All other variables in the expression are called free variables.

- **A formula that does not contain any free variables is a proposition and has a truth value**
Quantifier Rule

- Rule of inference

\[
\forall x \ P(x) \quad \frac{}{P(v)} \quad v \text{ is a value in assumed domain}
\]
Quantifier Rule

- Rule of inference

\[
\frac{\forall x \ P(x)}{P(v)} \quad v \text{ is a value in assumed domain}
\]

From \( \forall x \ P(x) \) is true infer that \( P(v) \) is true for any value \( v \) in the assumed domain

- \( \neg(\forall x \ P(x)) \leftrightarrow \exists x \ \neg P(x) \quad \neg(\exists x \ P(x)) \leftrightarrow \forall x \ \neg P(x) \)
Quantifier Rule

- Rule of inference

\[
\frac{\forall x \ P(x)}{P(v)}\quad v \text{ is a value in assumed domain}
\]

From \(\forall x \ P(x)\) is true infer that \(P(v)\) is true for any value \(v\) in the assumed domain

- \(\neg(\forall x \ P(x)) \iff \exists x \ \neg P(x)\quad \neg(\exists x \ P(x)) \iff \forall x \ \neg P(x)\)

It is not the case that for all \(x\) \(P(x)\) if, and only if, \(P(x)\) is not true for some \(x\)
Quantifier Rule

- Rule of inference

\[
\frac{\forall x P(x)}{P(v)} \quad \text{\(v\) is a value in assumed domain}
\]

From \(\forall x P(x)\) is true infer that \(P(v)\) is true for any value \(v\) in the assumed domain.

- \(\neg(\forall x P(x)) \leftrightarrow \exists x \neg P(x)\)
  \(\neg(\exists x P(x)) \leftrightarrow \forall x \neg P(x)\)

It is not the case that for all \(x\) \(P(x)\) if, and only if, \(P(x)\) is not true for some \(x\).

- We always assume that a domain is nonempty.
Our example

- From *All humans are mortal* and *Socrates is human* derive *Socrates is mortal*
Our example

- From **All humans are mortal** and **Socrates is human** derive **Socrates is mortal**
- \( H(x) \) is “\( x \) is human”; \( M(x) \) is “\( x \) is mortal”
Our example

- From *All humans are mortal* and *Socrates is human* derive *Socrates is mortal*
- $H(x)$ is “$x$ is human”; $M(x)$ is “$x$ is mortal”
- All humans are mortal  \( \forall x \ (H(x) \rightarrow M(x)) \)
Our example

- From *All humans are mortal* and *Socrates is human* derive *Socrates is mortal*
- $H(x)$ is “$x$ is human”; $M(x)$ is “$x$ is mortal”
- All humans are mortal  $\forall x \ (H(x) \to M(x))$
- Socrates is human  $H(Socrates)$
Our example

- From *All humans are mortal* and *Socrates is human* derive *Socrates is mortal*
- \( H(x) \) is “\( x \) is human”; \( M(x) \) is “\( x \) is mortal”
- All humans are mortal \( \forall x \ (H(x) \rightarrow M(x)) \)
- Socrates is human \( H(Socrates) \)
- How do we get \( M(Socrates) \)?
Our example

- From All humans are mortal and Socrates is human derive Socrates is mortal
- $H(x)$ is “$x$ is human”; $M(x)$ is “$x$ is mortal”
- All humans are mortal $\forall x \ (H(x) \rightarrow M(x))$
- Socrates is human $H(Socrates)$
- How do we get $M(Socrates)$?
- From $\forall x \ (H(x) \rightarrow M(x))$ we derive $H(Socrates) \rightarrow M(Socrates)$
Our example

- From **All humans are mortal** and **Socrates is human** derive **Socrates is mortal**
- $H(x)$ is “$x$ is human”; $M(x)$ is “$x$ is mortal”
- All humans are mortal  $\forall x \ (H(x) \rightarrow M(x))$
- Socrates is human  $H(Socrates)$
- How do we get $M(Socrates)$?
- From $\forall x \ (H(x) \rightarrow M(x))$ we derive $H(Socrates) \rightarrow M(Socrates)$
- By propositional reasoning, $(p \rightarrow q$ and $p)$ implies $q$
Our example

- From *All humans are mortal* and *Socrates is human* derive *Socrates is mortal*
- $H(x)$ is “$x$ is human”; $M(x)$ is “$x$ is mortal”
- All humans are mortal  $\forall x (H(x) \rightarrow M(x))$
- Socrates is human  $H(Socrates)$
- How do we get $M(Socrates)$?
- From $\forall x (H(x) \rightarrow M(x))$ we derive $H(Socrates) \rightarrow M(Socrates)$
- By propositional reasoning, $(p \rightarrow q$ and $p)$ implies $q$
  So, $H(Socrates) \rightarrow M(Socrates)$ and $H(Socrates)$ implies $M(Socrates)$
Proving $\forall x \ P(x)$

- **Rule of inference**

\[
\frac{P(c)}{\forall x \ P(x)} \quad c \text{ is an arbitrary element of domain}
\]
Proving $\forall x \, P(x)$

- **Rule of inference**
  \[
  \begin{align*}
  P(c) \\
  \forall x \, P(x)
  \end{align*}
  \]
  $c$ is an arbitrary element of domain

- **Example**: if $n$ is an odd integer then $n^2$ is odd
Proving $\forall x \ P(x)$

- Rule of inference

$$\frac{P(c)}{\forall x \ P(x)} \quad c \text{ is an arbitrary element of domain}$$

- Example: if $n$ is an odd integer then $n^2$ is odd
- First, notice the quantifier is implicit
Proving $\forall x \ P(x)$

- **Rule of inference**

\[
\begin{align*}
\frac{P(c)}{\forall x \ P(x)} \quad & \text{c is an arbitrary element of domain} \\
\end{align*}
\]

- **Example**: if $n$ is an odd integer then $n^2$ is odd

- **First, notice the quantifier is implicit**

- **Let** $P(n)$ be “$n$ is odd” and $Q(n)$ be “the square of $n$ is odd”
Proving $\forall x \ P(x)$

- Rule of inference

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\frac{P(c)}{\forall x \ P(x)} \quad \text{c is an arbitrary element of domain}
\]

- Example: if $n$ is an odd integer then $n^2$ is odd
- First, notice the quantifier is implicit
- Let $P(n)$ be “$n$ is odd” and $Q(n)$ be “the square of $n$ is odd”
- So is: $\forall x \ (P(x) \rightarrow Q(x))$
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain.
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
- Use the definition/properties of $P(n)$, $n$ is odd
Direct proof of $\forall x (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
- Use the definition/properties of $P(n)$, $n$ is odd
- $P(n)$ provided that for some $k$, $n = 2k + 1$
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
- That is, assume $n$ is odd, then show $n^2$ is odd
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- $P(n)$ provided that for some $k$, $n = 2k + 1$
- So $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$
Direct proof of $\forall x \ (P(x) \rightarrow Q(x))$

- Assume $n$ is an arbitrary element of the domain
- Prove that $P(n) \rightarrow Q(n)$
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- $P(n)$ provided that for some $k$, $n = 2k + 1$
- So $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$
- $n^2$ has the form for some $m$, $n^2 = 2m + 1$; so $Q(n)$
Any odd integer is the difference of two squares
Proving $\forall x \ (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$

Assume $c$ is an arbitrary element of the domain
Prove that $\neg B(c) \rightarrow \neg A(c)$
That is, assume $\neg B(c)$ then show $\neg A(c)$
Use the definition/properties of $\neg B(c)$
Proving $\forall x (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$
- So, $\forall x (A(x) \rightarrow B(x)) \iff \forall x (\neg B(x) \rightarrow \neg A(x))$
Proving $\forall x \ (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$
- So, $\forall x \ (A(x) \rightarrow B(x)) \iff \forall x \ (\neg B(x) \rightarrow \neg A(x))$
- Assume $c$ is an arbitrary element of the domain
Proving $\forall x \ (A(x) \rightarrow B(x))$ by contraposition

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- So, $\forall x (A(x) \rightarrow B(x)) \iff \forall x (\neg B(x) \rightarrow \neg A(x))$
- Assume $c$ is an arbitrary element of the domain
- Prove that $\neg B(c) \rightarrow \neg A(c)$
- That is, assume $\neg B(c)$ then show $\neg A(c)$
Proving $\forall x \ (A(x) \rightarrow B(x))$ by contraposition

- Uses equivalence of $(p \rightarrow q)$ and $(\neg q \rightarrow \neg p)$
- So, $\forall x \ (A(x) \rightarrow B(x)) \iff \forall x \ (\neg B(x) \rightarrow \neg A(x))$
- Assume $c$ is an arbitrary element of the domain
- Prove that $\neg B(c) \rightarrow \neg A(c)$
- That is, assume $\neg B(c)$ then show $\neg A(c)$
- Use the definition/properties of $\neg B(c)$
if \( x + y \) is even, then \( x \) and \( y \) have the same parity
if $x + y$ is even, then $x$ and $y$ have the same parity

**Proof** Let $n, m \in \mathbb{Z}$ be arbitrary. We will prove that if $n$ and $m$ do not have the same parity then $n + m$ is odd. Without loss of generality we assume that $n$ is odd and $m$ is even, that is $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $m = 2\ell$ for some $\ell \in \mathbb{Z}$. But then $n + m = 2k + 1 + 2\ell = 2(k + \ell) + 1$. And thus $n + m$ is odd. Now by equivalence of a statement with it contrapositive derive that if $n + m$ is even, then $n$ and $m$ have the same parity.
If $n = ab$ where $a, b$ are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$
Proof by contradiction

- Want to prove that \( p \) is true
Proof by contradiction

- Want to prove that \( p \) is true
- Assume \( \neg p \)
Proof by contradiction

- Want to prove that $p$ is true
- Assume $\neg p$
- Derive both $q$ and $\neg q$ (a contradiction equivalent to False)
Proof by contradiction

- Want to prove that $p$ is true
- Assume $\neg p$
- Derive both $q$ and $\neg q$ (a contradiction equivalent to False)
- Therefore, $\neg\neg p$ which is equivalent to $p$
\[ \sqrt{2} \text{ is irrational} \]
The square root of 2 is irrational

Proof Assume towards a contradiction that \( \sqrt{2} \) is rational, that is there are integers \( a \) and \( b \) with no common factor other than 1, such that \( \sqrt{2} = a/b \). In that case \( 2 = a^2/b^2 \). Multiplying both sides by \( b^2 \), we have \( a^2 = 2b^2 \). Since \( b \) is an integer, so is \( b^2 \), and thus \( a^2 \) is even. As we saw previously this implies that \( a \) is even, that is there is an integer \( c \) such that \( a = 2c \). Hence \( 2b^2 = 4c^2 \), hence \( b^2 = 2c^2 \). Now, since \( c \) is an integer, so is \( c^2 \), and thus \( b^2 \) is even. Again, we can conclude that \( b \) is even. Thus \( a \) and \( b \) have a common factor 2, contradicting the assertion that \( a \) and \( b \) have no common factor other than 1. This shows that the original assumption that \( \sqrt{2} \) is rational is false, and that \( \sqrt{2} \) must be irrational.
There are infinitely many primes

Lemma Every natural number greater than one is either prime or it has a prime divisor

Proof Suppose towards a contradiction that there are only finitely many primes $p_1, p_2, p_3, \ldots, p_k$. Consider the number $q = p_1 p_2 p_3 \ldots p_k + 1$, the product of all the primes plus one. By hypothesis $q$ cannot be prime because it is strictly larger than all the primes. Thus, by the lemma, it has a prime divisor, $p$. Because $p_1, p_2, p_3, \ldots, p_k$ are all the primes, $p$ must be equal to one of them, so $p$ is a divisor of their product. So we have that $p$ divides $p_1 p_2 p_3 \ldots p_k$, and $p$ divides $q$, but that means $p$ divides their difference, which is 1. Therefore $p \leq 1$. Contradiction. Therefore there are infinitely many primes.
There are infinitely many primes

**Lemma** Every natural number greater than one is either prime or it has a prime divisor
There are infinitely many primes

**Lemma** Every natural number greater than one is either prime or it has a prime divisor

**Proof** Suppose towards a contradiction that there are only finitely many primes $p_1, p_2, p_3, \ldots, p_k$. Consider the number $q = p_1p_2p_3\ldots p_k + 1$, the product of all the primes plus one. By hypothesis $q$ cannot be prime because it is strictly larger than all the primes. Thus, by the lemma, it has a prime divisor, $p$. Because $p_1, p_2, p_3, \ldots, p_k$ are all the primes, $p$ must be equal to one of them, so $p$ is a divisor of their product. So we have that $p$ divides $p_1p_2p_3\ldots p_k$, and $p$ divides $q$, but that means $p$ divides their difference, which is 1. Therefore $p \leq 1$. Contradiction. Therefore there are infinitely many primes.
Proof by cases

- To prove a conditional statement of the form

\[(p_1 \lor \cdots \lor p_k) \rightarrow q\]

- Use the tautology

\[((p_1 \lor \cdots \lor p_k) \rightarrow q) \iff ((p_1 \rightarrow q) \land \cdots \land (p_k \rightarrow q))\]

- Each of the implications \(p_i \rightarrow q\) is a case
If $n$ is an integer then $n^2 \geq n$
Proof of $\exists x \ P(x)$

Rule of inference

$$\frac{P(v)}{\exists x \ P(x)} \quad v \text{ is a value in the domain}$$
Proof of $\exists x \ P(x)$

Rule of inference

$$\frac{P(v)}{\exists x \ P(x)} \quad v \text{ is a value in the domain}$$

Constructive proof: exhibit an actual witness $w$ from the domain such that $P(w)$ is true. Therefore, $\exists x \ P(x)$
There exists a positive integer that can be written as the sum of cubes of positive integers in two different ways.

1729 is such a number because

\[ 10^3 + 9^3 = 1729 = 12^3 + 1^3 \]
There exists a positive integer that can be written as the sum of cubes of positive integers in two different ways

- 1729 is such a number because
  \[10^3 + 9^3 = 1729 = 12^3 + 1^3\]
There exists a positive integer that can be written as the sum of cubes of positive integers in two different ways

- 1729 is such a number because
  \[ 10^3 + 9^3 = 1729 = 12^3 + 1^3 \]
Nonconstructive proof of $\exists x \ P(x)$

Show that there must be a value $v$ such that $P(v)$ is true.

But we don't know what this value $v$ is.
Nonconstructive proof of $\exists x \, P(x)$

- Show that there must be a value $v$ such that $P(v)$ is true
Nonconstructive proof of $\exists x \ P(x)$

- Show that there must be a value $v$ such that $P(v)$ is true
- But we don’t know what this value $v$ is
There exist irrational numbers $x$ and $y$ such that $x^y$ is rational.
There exist irrational numbers \( x \) and \( y \) such that \( x^y \) is rational

Proof We need only prove the existence of at least one example. Consider the case \( x = \sqrt{2} \) and \( y = \sqrt{2} \). We distinguish two cases:

Case \( \sqrt{2}^{\sqrt{2}} \) is rational. In that case we have shown that for the irrational numbers \( x = y = \sqrt{2} \), we have that \( x^y \) is rational

Case \( \sqrt{2}^{\sqrt{2}} \) is irrational. In that case consider \( x = \sqrt{2}^{\sqrt{2}} \) and \( y = \sqrt{2} \). We then have that

\[
x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2
\]

But since 2 is rational, we have shown that for \( x = \sqrt{2}^{\sqrt{2}} \) and \( y = \sqrt{2} \), we have that \( x^y \) is rational

We have thus shown that in any case there exist some irrational numbers \( x \) and \( y \) such that \( x^y \) is rational
Disproving $\forall x \ P(x)$ with a counter-example

- $\neg \forall x \ P(x)$ is equivalent to $\exists x \ \neg P(x)$

So, $w$ is a counterexample to the assertion $\forall x \ P(x)$.
Disproving $\forall x \ P(x)$ with a counter-example

- $\neg \forall x \ P(x)$ is equivalent to $\exists x \ \neg P(x)$
- To establish that $\neg \forall x \ P(x)$ is true find a $w$ such that $P(w)$ is false
Disproving $\forall x \ P(x)$ with a counter-example

- $\neg \forall x \ P(x)$ is equivalent to $\exists x \ \neg P(x)$
- To establish that $\neg \forall x \ P(x)$ is true find a $w$ such that $P(w)$ is false
- So, $w$ is a counterexample to the assertion $\forall x \ P(x)$
Every positive integer is the sum of the squares of 3 integers

The integer 7 is a counterexample. So the claim is false.
Every positive integer is the sum of the squares of 3 integers

The integer 7 is a counterexample. So the claim is false
Nested quantifiers

- Every real number has an inverse w.r.t addition (domain $\mathbb{R}$)

$$\forall x \exists y (x + y = 0)$$
Nested quantifiers

- Every real number has an inverse w.r.t addition (domain $\mathbb{R}$)
  \[
  \forall x \exists y (x + y = 0)
  \]

- Every real number except zero has an inverse w.r.t multiplication
  \[
  \forall x (x \neq 0 \rightarrow \exists y (x \times y = 1))
  \]
Nested quantifiers

- Every real number has an inverse w.r.t addition (domain $\mathbb{R}$)
  \[ \forall x \ \exists y \ (x + y = 0) \]

- Every real number except zero has an inverse w.r.t multiplication
  \[ \forall x \ (x \neq 0 \rightarrow \exists y \ (x \times y = 1)) \]

- $\lim_{x \to a} f(x) = b$
  \[ \forall \epsilon \ \exists \delta \ \forall x \ (0 < |x - a| < \delta \rightarrow |f(x) - b| < \epsilon) \]
Nested quantifiers

- Every real number has an inverse w.r.t addition (domain \( \mathbb{R} \))
  \[
  \forall x \exists y \ (x + y = 0)
  \]

- Every real number except zero has an inverse w.r.t multiplication
  \[
  \forall x \ (x \neq 0 \rightarrow \exists y \ (x \times y = 1))
  \]

- \( \lim_{x \to a} f(x) = b \)
  \[
  \forall \epsilon \exists \delta \ \forall x \ (0 < |x - a| < \delta \rightarrow |f(x) - b| < \epsilon)
  \]

- \( \neg (\lim_{x \to a} f(x) = b) \)
  \[
  \exists \epsilon \ \forall \delta \ \exists x \ ((0 < |x - a| < \delta) \land (|f(x) - b| \geq \epsilon))
  \]