Discrete Mathematics & Mathematical Reasoning
Multiplicative Inverses and Some Cryptography

Colin Stirling

Informatics
Multiplicative inverses

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- $x = 8$ and $m = 15$. Then $x \cdot 2 = 16 \equiv 1 \pmod{15}$, so $2$ is a multiplicative inverse of $8 \pmod{15}$
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- $x = 8$ and $m = 15$. Then $x^2 = 16 \equiv 1 \pmod{15}$, so 2 is a multiplicative inverse of 8 $(\mod 15)$

- $x = 12$ and $m = 15$
  The sequence \{ $xa \mod m$ | $a = 0, 1, 2, \ldots$ \} is periodic, and takes on the values \{ 0, 12, 9, 6, 3 \}. So, 12 has no multiplicative inverse mod 15
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- Notice \( \gcd(8, 15) = 1 \) whereas \( \gcd(12, 15) = 3 \)
Multiplicative inverses mod $m$ when $\gcd(m, x) = 1$

**Theorem**

If $m, x$ are positive integers and $\gcd(m, x) = 1$ then $x$ has a multiplicative inverse mod $m$ (and it is unique mod $m$)

Proof. By Bézout's theorem there are $s$ and $t$ such that $sm + tx = 1 = \gcd(m, x)$.

So, $sm + tx \equiv 1 \pmod{m}$. As $sm \equiv 0 \pmod{m}$, so $tx \equiv 1 \pmod{m}$.

For uniqueness mod $m$. Assume $tx \equiv 1 \pmod{m}$ and $ux \equiv 1 \pmod{m}$.

Therefore, $tx \equiv ux \pmod{m}$. Since $\gcd(m, x) = 1$ it follows that $t \equiv u \pmod{m}$. 
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Chinese remainder theorem

Theorem

Let \( m_1, m_2, \ldots, m_n \) be pairwise relatively prime positive integers greater than 1 and \( a_1, a_2, \ldots, a_n \) be arbitrary integers. Then the system

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\begin{align*}
  x &\equiv a_1 \pmod{m_1} \\
  x &\equiv a_2 \pmod{m_2} \\
  \vdots \\
  x &\equiv a_n \pmod{m_n}
\end{align*}
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has a unique solution modulo \( m = m_1 m_2 \cdots m_n \)
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Proof.

In the book
Example

\[ x \equiv 2 \pmod{3} \]
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- \[ m = 3 \cdot 5 \cdot 7 = 105 \]
- \[ M_1 = 35 \text{ and } 2 \text{ is an inverse of } M_1 \pmod{3} \]

\[ x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 5 \cdot 15 \cdot 1 = 140 + 63 + 75 = 278 \equiv 68 \pmod{105} \]
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- \( m = 3 \cdot 5 \cdot 7 = 105 \)
- \( M_1 = 35 \) and 2 is an inverse of \( M_1 \) mod 3
- \( M_2 = 21 \) and 1 is an inverse of \( M_2 \) mod 5

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Fermat’s little theorem

**Theorem**

If $p$ is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$. Furthermore, for every integer $a$ we have $a^p \equiv a \pmod{p}$.
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Theorem

If $p$ is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$. Furthermore, for every integer $a$ we have $a^p \equiv a \pmod{p}$.

Proof.

Assume $p \nmid a$ and so, therefore, $\gcd(p, a) = 1$. Then $a, 2a, \ldots, (p - 1)a$ are not pairwise congruent modulo $p$; if $ia \equiv ja \pmod{p}$ because $\gcd(p, a) = 1$ then $i \equiv j \pmod{p}$ which is impossible. Therefore, each element $ja \pmod{p}$ is a distinct element in the set $\{1, \ldots, p - 1\}$. This means that the product $a \cdot 2a \cdot \cdots (p - 1)a \equiv 1 \cdot 2 \cdot \cdots p - 1 \pmod{p}$. Therefore, $(p - 1)!a^{p-1} \equiv (p - 1)! \pmod{p}$. Now because $\gcd(p, q) = 1$ for $1 \leq q \leq p - 1$ it follows that $a^{p-1} \equiv 1 \pmod{p}$. Therefore, also $a^p \equiv a \pmod{p}$ and when $p|a$ then clearly $a^p \equiv a \pmod{p}$. 
Computing the remainders modulo prime \( p \)

- Find \( 7^{222} \mod 11 \)

By Fermat's little theorem, we know that \( 7^{10} \equiv 1 \mod 11 \), and so \( (7^{10})^k \equiv 1 \mod 11 \) for every positive integer \( k \). Therefore, \( 7^{222} = 7^{22 \cdot 10 + 2} = (7^{10})^{22} \cdot 7^2 \equiv 1^{22} \cdot 49 \equiv 5 \mod 11 \). Hence, \( 7^{222} \mod 11 = 5 \).
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  $7^{222} = 7^{22 \cdot 10 + 2} = (7^{10})^{22} \cdot 7^2 \equiv 1^{22} \cdot 49 \equiv 5 \pmod{11}$. Hence, $7^{222} \mod 11 = 5$

- $2^{340} \equiv 1 \pmod{11}$ because $2^{10} \equiv 1 \pmod{11}$
Private key cryptography

- Bob wants to send Alice a secret message $M$

Alice sends Bob a private key $E_n$ (which has an inverse $D_n$)

Bob encrypts $M$ and sends $E_n(M)$

Alice decrypts $E_n(M)$, $D_n(E_n(M))$

Important property $D_n(E_n(M)) = M$

Alice and Bob share a secret which could be intercepted by a third party

Example use $E_n(p) = (p + 3) \mod 26$

What is WKLV LV D VHFSHW?
Private key cryptography

- Bob wants to send Alice a secret message M
- Alice sends Bob a private key En (which has an inverse De)

\[
\text{Bob encrypts } M \text{ and sends } En(M) \\
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Public key cryptography

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Public key cryptography

- Bob wants to send Alice a secret message $M$
- Without Alice and Bob sharing a secret

Alice sends Bob a public key $E_n$ (and keeps her inverse private key $D_e$ secret from everyone including Bob)

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The challenge: $D_e$ can't be feasibly computed from $E_n$; and given $E_n(M)$ one can't feasibly compute $M$
Public key cryptography

- Bob wants to send Alice a secret message M
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RSA Cryptosystem: Rivest, Shamir and Adleman

- Choose two distinct prime numbers $p$ and $q$
- Let $n = pq$ and $k = (p - 1)(q - 1)$
- Choose integer $e$ where $1 < e < k$ and $\gcd(e, k) = 1$
- $(n, e)$ is released as the public key
- Let $d$ be the multiplicative inverse of $e$ modulo $k$, so $de \equiv 1 \pmod{k}$
- $(n, d)$ is the private key and kept secret
RSA: encryption and decryption

Alice transmits her public key \((n, e)\) to Bob and keeps the private key \((n, d)\) secret

Encryption

Bob wishes to send message \(M\) to Alice

1. He turns \(M\) into integer \(m\), \(0 \leq m < n\), using an agreed-upon reversible protocol known as a padding scheme
2. He computes the ciphertext \(c\) corresponding to \(c = m^e \mod n\). (This can be done quickly)
3. Bob transmits \(c\) to Alice.

Decryption

Alice can recover \(m\) from \(c\)

1. Using her private key exponent \(d\) via computing \(m = c^d \mod n\)
2. Given \(m\), she can recover the original message \(M\) by reversing the padding scheme
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Example

- $n = 43 \cdot 59 = 2537$
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- \( \gcd(13, 42 \cdot 58) = 1 \), so public key is \((2537, 13)\)

\[ 1819 \mod 2537 = 2081 \quad \text{and} \quad 1415 \mod 2537 = 2182 \]

So encrypted message is 2081 2182
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- $n = 43 \cdot 59 = 2537$
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- $d = 937$ is inverse of 13 modulo $2436 = 42 \cdot 58$; private key $(2537, 937)$
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- Encrypt STOP as two blocks 1819 for ST and 1415 for OP (padding scheme)
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- So, \( 1819^{13} \mod 2537 = 2081 \) and \( 1415^{13} \mod 2537 = 2182 \)
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- So encrypted message is 2081 2182
RSA: correctness of decryption

Given that $c = m^e \mod n$, is $m = c^d \mod n$?

$$c^d = (m^e)^d \equiv m^{ed} \pmod{n}$$

By construction, $d$ and $e$ are each others multiplicative inverses modulo $k$, i.e. $ed \equiv 1 \pmod{k}$. Also $k = (p - 1)(q - 1)$. Thus

$ed - 1 = h(p - 1)(q - 1)$ for some integer $h$. We consider $m^{ed} \mod p$

If $p \nmid m$ then

$m^{ed} = m^{h(p-1)(q-1)} m = (m^{p-1})^{h(q-1)} m \equiv 1^{h(q-1)}m \equiv m \pmod{p}$ (by Fermat’s little theorem)

Otherwise $m^{ed} \equiv 0 \equiv m \pmod{p}$

Symmetrically, $m^{ed} \equiv m \pmod{q}$

Since $p, q$ are distinct primes, we have $m^{ed} \equiv m \pmod{pq}$. Since $n = pq$, we have $c^d = m^{ed} \equiv m \pmod{n}$