Discrete Mathematics & Mathematical Reasoning
Cardinality

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Informatics

Some slides based on ones by Myrto Arapinis and by Richard Mayr

Cardinality of Sets

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$\mathbb{N}$ and its subset Even $= \{2n \mid n \in \mathbb{N}\}$ have the same cardinality, because $f : \mathbb{N} \to$ Even where $f(n) = 2n$ is a bijection

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- A set is called countable iff it is either finite or countably infinite
- A set that is not countable is called uncountable

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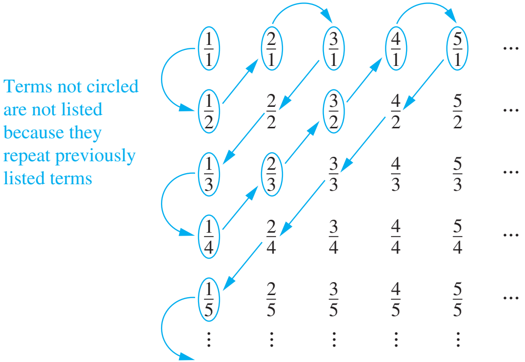

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- First define an (alphabetical) ordering on the symbols in Σ
  Show that the strings can be listed in a sequence
  - First single string ε of length 0
  - Then all strings of length 1 in lexicographic order
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  - etc

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The set of Java-programs is countable; a program is just a finite string

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- with the property $d_m = d(m)$ is the mth symbol (starting from 0)

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Let X be the set of infinite binary strings. For a contradiction assume that a bijection $f : \mathbb{N} \to X$ exists. So, f must be onto (surjective). Assume that $f(i) = d^i$ for $i \in \mathbb{N}$. So, $X = \{d^0, d^1, \ldots, d^m, \ldots\}$. Define the infinite binary string d as follows: $d_n = (d^n_n + 1) \bmod 2$. But for each m, $d \neq d^m$ because $d_m \neq d^m_m$. So, f is not a surjection. $\square$

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Similar argument shows that $\mathbb{R}$ via $[0, 1]$ is uncountable using infinite decimal strings (see book). "Most functions" are not computable!

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- Example $|(0,1)| = |(0,1]|$
- $|(0,1)| \leq |(0,1]|$ using identity function
- $|(0,1]| \leq |(0,1)|$ use $f(x) = x/2$ as $(0,1/2] \subset (0,1)$

Cantor's theorem

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Consider the injection $f : A \to \mathcal{P}(A)$ with $f(a) = \{a\}$ for any $a \in A$. Therefore, $|A| \leq |\mathcal{P}(A)|$. Next we show there is not a surjection $f : A \to \mathcal{P}(A)$. For a contradiction, assume that a surjection f exists. We define the set $B \subseteq A$: $B = \{x \in A \mid x \notin f(x)\}$. Since f is a surjection, there must exist an $a \in A$ s.t. $B = f(a)$. Now there are two cases:

1. If $a \in B$ then, by definition of B, $a \notin f(a) = B$. Contradiction
2. If $a \notin B$ then $a \notin f(a)$. Thus, by definition of B, $a \in B$. Contradiction

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