Discrete Mathematics & Mathematical Reasoning Sequences and Sums

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Slides based on ones by Myrto Arapinis

Colin Stirling (Informatics)

Discrete Mathematics (Section 2.4)

Today 1 / 14

Sequences

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2, 3, 5, 7, 11, 13, 17, 19, ... or a, b, c, d, ..., y, z

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Definition

A sequence over a set *S* is a function *f* from a subset of the integers (typically \mathbb{N} or \mathbb{Z}^+) to the set *S*. If the domain of *f* is finite then the sequence is finite

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Assuming $b_n = g(n)$, also written $b_0, b_1, b_2, ...$ or as $\{b_n\}_{n \in \mathbb{N}}$

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 with $c_n = 7 - 3n$

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• Example
$$\{c_n\}_{n\in\mathbb{N}}$$
 with $c_n = 7 - 3n$

where the initial elements a, the common ratio r and the common difference d are real numbers

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- The initial conditions specify the first elements of the sequence, before the recurrence relation applies
- A sequence is called a solution of a recurrence relation iff its terms satisfy the recurrence relation

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Answer is the Fibonacci sequence

$$\begin{cases} f(0) = 0 \\ f(1) = 1 \\ f(n) = f(n-1) + f(n-2) & \text{for } n \ge 2 \end{cases}$$

Yields the sequence 0, 1, 1, 2, 3, 5, 8, 13, ...

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- The guess can be proved correct by the method of induction (to be covered)

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$$\begin{array}{rrrr} a_2 &=& 2+3\\ a_3 &=& (2+3)+3=2+3\cdot 2\\ a_4 &=& (2+2\cdot 3)+3=2+3\cdot 3 \end{array}$$

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$$\begin{array}{rcl} a_2 &=& 2+3\\ a_3 &=& (2+3)+3=2+3\cdot 2\\ a_4 &=& (2+2\cdot 3)+3=2+3\cdot 3\\ &\vdots\\ a_n &=& a_{n-1}+3=(2+3\cdot (n-2))+3=2+3\cdot (n-1) \end{array}$$

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Iterative solution - working downward

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:
= $a_2 + 3(n-2) = (a_1 + 3) + 3 \cdot (n-2) = 2 + 3 \cdot (n-1)$

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- $P_{20} = (1.03)^{20} \, 1000 = 1,806$

Common sequences

TABLE 1 Some Useful Sequences.		
nth Term	First 10 Terms	
n ²	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,	
n ³	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,	
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,	
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,	
3 ⁿ	$3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, \ldots$	
n!	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,	
fn	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,	

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Summations

Given a sequence $\{a_n\}$, the sum of terms $a_m, a_{m+1}, \ldots, a_\ell$ is

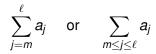
 $a_m + a_{m+1} + \ldots + a_\ell$

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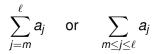
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The variable *j* is called the index of summation

More generally for an index set S

$$\sum_{j\in S} a_j$$

Useful summation formulas

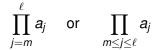
TABLE 2 Some Useful Summation Formulae.		
Sum	Closed Form	
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$	
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$	
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$	
$\sum_{k=0}^{\infty} x^k, x < 1$	$\frac{1}{1-x}$	
$\sum_{k=1}^{\infty} k x^{k-1}, x < 1$	$\frac{1}{(1-x)^2}$	

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Products

Given a sequence $\{a_n\}$, the product of terms $a_m, a_{m+1}, \ldots, a_\ell$ is

 $a_m \cdot a_{m+1} \cdot \ldots \cdot a_\ell$



More generally for a finite index set S one writes

 $\prod_{j\in S} a_j$