Discrete Mathematics & Mathematical Reasoning
Sequences and Sums

Colin Stirling

Informatics

Slides based on ones by Myrto Arapinis

Sequences

Sequences are ordered lists of elements

$2, 3, 5, 7, 11, 13, 17, 19, \ldots$ or $a, b, c, d, \ldots, y, z$

Sequences

Sequences are ordered lists of elements

$2, 3, 5, 7, 11, 13, 17, 19, \ldots$ or $a, b, c, d, \ldots, y, z$

Definition

A sequence over a set S is a function f from a subset of the integers (typically $\mathbb{N}$ or $\mathbb{Z}^+$) to the set S. If the domain of f is finite then the sequence is finite

Examples

$f : \mathbb{Z}^+ \to \mathbb{Q}$ is $f(n) = 1/n$ defines the sequence

Examples

$f : \mathbb{Z}^+ \to \mathbb{Q}$ is $f(n) = 1/n$ defines the sequence

$$1,\ 1/2,\ 1/3,\ 1/4, \ldots$$

Examples

$f : \mathbb{Z}^+ \rightarrow \mathbb{Q}$ is $f(n) = 1/n$ defines the sequence

$$1,\ 1/2,\ 1/3,\ 1/4, \ldots$$

Assuming $a_n = f(n)$, the sequence is also written a_1, a_2, a_3, ...

or as $\{a_n\}_{n \in \mathbb{Z}^+}$

Examples

$f : \mathbb{Z}^+ \rightarrow \mathbb{Q}$ is $f(n) = 1/n$ defines the sequence

$$1,\ 1/2,\ 1/3,\ 1/4, \ldots$$

Assuming $a_n = f(n)$, the sequence is also written a_1, a_2, a_3, ...

or as $\{a_n\}_{n \in \mathbb{Z}^+}$

$g : \mathbb{N} \rightarrow \mathbb{N}$ is $g(n) = n^2$ defines the sequence

$$0,\ 1,\ 4,\ 9, \ldots$$

Examples

$f : \mathbb{Z}^+ \to \mathbb{Q}$ is $f(n) = 1/n$ defines the sequence

$$1,\ 1/2,\ 1/3,\ 1/4, \ldots$$

Assuming $a_n = f(n)$, the sequence is also written a_1, a_2, a_3, ...

or as $\{a_n\}_{n \in \mathbb{Z}^+}$

$g : \mathbb{N} \to \mathbb{N}$ is $g(n) = n^2$ defines the sequence

$$0,\ 1,\ 4,\ 9, \ldots$$

Assuming $b_n = g(n)$, also written b_0, b_1, b_2, $\ldots$

or as $\{b_n\}_{n \in \mathbb{N}}$

Geometric and arithmetic progressions

- A geometric progression is a sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n,\ \ldots$$

Geometric and arithmetic progressions

- A geometric progression is a sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n,\ \ldots$$

- Example $\{b_n\}_{n\in\mathbb{N}}$ with $b_n = (-1)^n$

Geometric and arithmetic progressions

- A geometric progression is a sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n,\ \ldots$$

- Example $\{b_n\}_{n\in\mathbb{N}}$ with $b_n = (-1)^n$

- An arithmetic progression is a sequence of the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+nd,\ \ldots$$

Geometric and arithmetic progressions

- A geometric progression is a sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n,\ \ldots$$

- Example $\{b_n\}_{n\in\mathbb{N}}$ with $b_n = (-1)^n$

- An arithmetic progression is a sequence of the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+nd,\ \ldots$$

- Example $\{c_n\}_{n\in\mathbb{N}}$ with $c_n = 7 - 3n$

Geometric and arithmetic progressions

- A geometric progression is a sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n,\ \ldots$$

- Example $\{b_n\}_{n\in\mathbb{N}}$ with $b_n = (-1)^n$

- An arithmetic progression is a sequence of the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+nd,\ \ldots$$

- Example $\{c_n\}_{n\in\mathbb{N}}$ with $c_n = 7-3n$

where the initial elements a, the common ratio r and the common difference d are real numbers

Recurrence relations

Definition

A recurrence relation for $\{a_n\}_{n\in\mathbb{N}}$ is an equation that expresses a_n in terms of one or more of the elements $a_0, a_1, \ldots, a_{n-1}$

Recurrence relations

Definition

A recurrence relation for $\{a_n\}_{n\in\mathbb{N}}$ is an equation that expresses a_n in terms of one or more of the elements $a_0, a_1, \ldots, a_{n-1}$

- Typically the recurrence relation expresses a_n in terms of just a fixed number of previous elements (such as $a_n = g(a_{n-1}, a_{n-2})$)

Recurrence relations

Definition

A recurrence relation for $\{a_n\}_{n\in\mathbb{N}}$ is an equation that expresses a_n in terms of one or more of the elements $a_0, a_1, \ldots, a_{n-1}$

- Typically the recurrence relation expresses a_n in terms of just a fixed number of previous elements (such as $a_n = g(a_{n-1}, a_{n-2})$)
- The initial conditions specify the first elements of the sequence, before the recurrence relation applies

Recurrence relations

Definition

A recurrence relation for $\{a_n\}_{n \in \mathbb{N}}$ is an equation that expresses a_n in terms of one or more of the elements $a_0, a_1, \ldots, a_{n-1}$

- Typically the recurrence relation expresses a_n in terms of just a fixed number of previous elements (such as $a_n = g(a_{n-1}, a_{n-2})$)
- The initial conditions specify the first elements of the sequence, before the recurrence relation applies
- A sequence is called a solution of a recurrence relation iff its terms satisfy the recurrence relation

Rabbits and Fibonacci sequence

A young pair of rabbits (one of each sex) is placed on an island

Rabbits and Fibonacci sequence

A young pair of rabbits (one of each sex) is placed on an island

A pair of rabbits does not breed until they are 2 months old. After they are 2 months old each pair produces another pair each month

Rabbits and Fibonacci sequence

A young pair of rabbits (one of each sex) is placed on an island

A pair of rabbits does not breed until they are 2 months old. After they are 2 months old each pair produces another pair each month

Find a recurrence relation for number of rabbits after n months assuming no rabbits die

Rabbits and Fibonacci sequence

A young pair of rabbits (one of each sex) is placed on an island

A pair of rabbits does not breed until they are 2 months old. After they are 2 months old each pair produces another pair each month

Find a recurrence relation for number of rabbits after n months assuming no rabbits die

Answer is the Fibonacci sequence

$$\begin{cases} f(0) = 0 \\ f(1) = 1 \\ f(n) = f(n-1) + f(n-2) \quad \text{for } n \geq 2 \end{cases}$$

Yields the sequence 0, 1, 1, 2, 3, 5, 8, 13, ...

Solving recurrence relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called solving the recurrence relation

Solving recurrence relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called solving the recurrence relation
- Such a formula is called a closed formula

Solving recurrence relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called solving the recurrence relation
- Such a formula is called a closed formula
- Various more advanced methods for solving recurrence relations are covered in Chapter 8 of the book (not part of this course)

Solving recurrence relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called solving the recurrence relation
- Such a formula is called a closed formula
- Various more advanced methods for solving recurrence relations are covered in Chapter 8 of the book (not part of this course)
- Here we illustrate by example the method of iteration in which we need to guess the formula

Solving recurrence relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called solving the recurrence relation
- Such a formula is called a closed formula
- Various more advanced methods for solving recurrence relations are covered in Chapter 8 of the book (not part of this course)
- Here we illustrate by example the method of iteration in which we need to guess the formula
- The guess can be proved correct by the method of induction (to be covered)

Iterative solution - working upwards

Forward substitution

Iterative solution - working upwards

Forward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

Iterative solution - working upwards

Forward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$a_2 = 2 + 3$$

Iterative solution - working upwards

Forward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$\begin{aligned} a_2 &= 2 + 3 \\ a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2 \end{aligned}$$

Iterative solution - working upwards

Forward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$
\begin{aligned}
a_2 &= 2 + 3 \\
a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2 \\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3
\end{aligned}
$$

Iterative solution - working upwards

Forward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$
\begin{aligned}
a_2 &= 2 + 3 \\
a_3 &= (2+3) + 3 = 2 + 3 \cdot 2 \\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \\
&\vdots \\
a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3 \cdot (n-1)
\end{aligned}
$$

Iterative solution - working downward

Backward substitution

Iterative solution - working downward

Backward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$a_n = a_{n-1} + 3$$

Iterative solution - working downward

Backward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \end{aligned}$$

Iterative solution - working downward

Backward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\ &= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \end{aligned}$$

Iterative solution - working downward

Backward substitution

$a_n = a_{n-1} + 3$ for $n \geq 2$ with $a_1 = 2$

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\vdots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3 \cdot (n-2) = 2 + 3 \cdot (n-1)
\end{aligned}
$$

Compound interest

- Suppose a person deposits £1000 in a savings account yielding 3% per year with interest compounded annually. How much is in the account after 20 years?

Compound interest

- Suppose a person deposits £1000 in a savings account yielding 3% per year with interest compounded annually. How much is in the account after 20 years?
- Let P_n denote amount after n years

Compound interest

- Suppose a person deposits £1000 in a savings account yielding 3% per year with interest compounded annually. How much is in the account after 20 years?
- Let P_n denote amount after n years
- $P_n = P_{n-1} + 0.03\, P_{n-1} \;=\; (1.03)P_{n-1}$

Compound interest

- Suppose a person deposits £1000 in a savings account yielding 3% per year with interest compounded annually. How much is in the account after 20 years?
- Let P_n denote amount after n years
- $P_n = P_{n-1} + 0.03\,P_{n-1} \;=\; (1.03)P_{n-1}$
- The initial condition $P_0 = 1000$.

Compound interest

- Suppose a person deposits £1000 in a savings account yielding 3% per year with interest compounded annually. How much is in the account after 20 years?
- Let P_n denote amount after n years
- $P_n = P_{n-1} + 0.03\, P_{n-1} \;\; = \;\; (1.03) P_{n-1}$
- The initial condition $P_0 = 1000$.
- $P_1 = (1.03)\, P_0, \ldots, \quad P_n = (1.03) P_{n-1} \;\; = (1.03)^n P_0$

Compound interest

- Suppose a person deposits £1000 in a savings account yielding 3% per year with interest compounded annually. How much is in the account after 20 years?
- Let P_n denote amount after n years
- $P_n = P_{n-1} + 0.03\, P_{n-1} \quad = \quad (1.03)P_{n-1}$
- The initial condition $P_0 = 1000$.
- $P_1 = (1.03)\, P_0, \ldots, \quad P_n = (1.03)P_{n-1} \quad = (1.03)^n P_0$
- $P_{20} = (1.03)^{20}\, 1000 = 1,806$

Common sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Summations

Given a sequence $\{a_n\}$, the sum of terms a_m, a_{m+1}, $\ldots$, a_ℓ is

$$a_m + a_{m+1} + \ldots + a_\ell$$

Summations

Given a sequence $\{a_n\}$, the sum of terms a_m, a_{m+1}, $\ldots$, a_ℓ is

$$a_m + a_{m+1} + \ldots + a_\ell$$

$$\sum_{j=m}^{\ell} a_j \quad \text{or} \quad \sum_{m \leq j \leq \ell} a_j$$

Summations

Given a sequence $\{a_n\}$, the sum of terms $a_m, a_{m+1}, \ldots, a_\ell$ is

$$a_m + a_{m+1} + \ldots + a_\ell$$

$$\sum_{j=m}^{\ell} a_j \quad \text{or} \quad \sum_{m \le j \le \ell} a_j$$

The variable j is called the index of summation

More generally for an index set S

$$\sum_{j \in S} a_j$$

Useful summation formulas

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \; (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Products

Given a sequence $\{a_n\}$, the product of terms $a_m, a_{m+1}, \ldots, a_\ell$ is

$$a_m \cdot a_{m+1} \cdot \ldots \cdot a_\ell$$

$$\prod_{j=m}^{\ell} a_j \quad \text{or} \quad \prod_{m \le j \le \ell} a_j$$

More generally for a finite index set S one writes

$$\prod_{j \in S} a_j$$