Sequences, sums, cardinality[1]

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[1]Slides mainly borrowed from Richard Mayr

Sequences

Sequences are ordered lists of elements, *e.g.* 2, 3, 5, 7, 11, 13, 17, 19, ... or a, b, c, d, ...

Definition

A sequence over a set S is a function f from a subset of the integers (typically $\mathbb{N}$ or $\mathbb{N} - \{0\}$) to the set S. If the domain of f is finite then the sequence is finite

Example Let $f : \mathbb{N} - \{0\} \to \mathbb{Q}$ be defined by $f(n) \stackrel{\text{def}}{=} 1/n$. This defines the sequence

$$1,\ 1/2,\ 1/3,\ 1/4, \ldots$$

Let $a_n = f(n)$. Then the sequence is also written as a_1, a_2, a_3, ... or as $\{a_n\}_{n \in \mathbb{N} - \{0\}}$

Geometric vs. Arithmetic progression

- A geometric progression is a sequence of the form

  $$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n,\ \ldots$$

  where both the initial element a and the common ratio r are real numbers

- An arithmetic progression is a sequence of the form

  $$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+nd,\ \ldots$$

  where both the initial element a and the common difference d are real numbers

Recurrence relations

Definition

A recurrence relation for the sequence $\{a_n\}_{n \in \mathbb{N}}$ is an equation that expresses a_n in terms of (one or more of) the previous elements a_0, a_1, . . . , a_{n-1} of the sequence

- Typically the recurrence relation expresses a_n in terms of just a fixed number of previous elements, *e.g.* $a_n = g(_{an-1}, a_{n-2}) = 2a_{n-1} + a_{n-2} + 7$
- The initial conditions specify the first elements of the sequence, before the recurrence relation applies
- A sequence is called a solution of a recurrence relation iff its terms satisfy the recurrence relation

Example Let $a_0 = 2$ and $a_n = a_{n-1} + 3$ for $n \geq 1$.Then $a_1 = 5$, $a_2 = 8$, $a_3 = 11$, *etc.* Generally the solution is $f(n) = 2 + 3n$

Fibonacci sequence

The Fibonacci sequence is described by the following linear recurrence relation

$$\begin{cases} f(0) & = & 0 \\ f(1) & = & 1 \\ f(n) & = & f(n-1) + f(n-2) \quad \text{for } n \geq 2 \end{cases}$$

You obtain the sequence 0, 1, 1, 2, 3, 5, 8, 13, . . .

How to solve general recurrence with $f(0) = a$, $f(1) = b$, $f(n) = cf(n-1) + df(n-2)$?
Linear algebra. Matrix multiplication. Base transforms. Diagonal form., *etc*

Solving recurrence relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called solving the recurrence relation
- Such a formula is called a closed formula
- Various methods for solving recurrence relations will be covered later in the course where recurrence relations will be studied in greater depth
- Here we illustrate by example the method of iteration in which we need to guess the formula
- The guess can be proved correct by the method of induction

Iterative solution - Example 1

Method 1: Working upward, forward substitution
Let a_n be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n \geq 2$ and suppose that $a_1 = 2$

$$
\begin{aligned}
a_2 &= 2 + 3 \\
a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2 \\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \\
\ldots \\
a_n &= an - 1 + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3 \cdot (n - 1)
\end{aligned}
$$

Iterative solution - Example 2

Method 2: Working downward, backward substitution
Let a_n be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n \geq 2$ and suppose that $a_1 = 2$

$$\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&= \ldots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3 \cdot (n-2) = 2 + 3 \cdot (n-1)
\end{aligned}$$

Common sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$
n^3	$1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, \ldots$
n^4	$1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, \ldots$
2^n	$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$
3^n	$3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, \ldots$
$n!$	$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, \ldots$
f_n	$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Summations

Given a sequence $\{a_n\}$. The sum of the terms a_m, a_{m+1}, ..., a_ℓ is written as

$$a_m + a_{m+1} + \ldots + a_\ell$$

$$\sum_{j=1}^{\ell} a_j$$

$$\sum_{m \leq j \leq \ell} a_j$$

The variable j is called the index of summation. It runs through all the integers starting with its lower limit m and ending with its upper limit ℓ. More generally for an index set S one writes

$$\sum_{j \in S} a_j$$

Useful summation formulas

TABLE 2 Some Useful Summation Formulae.	
Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \; (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Products

Given a sequence $\{a_n\}$. The sum of the terms a_m, a_{m+1}, ..., a_ℓ is written as

$$a_m * a_{m+1} * \ldots * a_\ell$$

$$\prod_{j=1}^{\ell} a_j$$

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- How many sequences over S of length $\leq n$ are there?
  Answer: Sum over the (non-overlapping!) cases of length $j = 0, 1, 2, \ldots, n$
  $$\sum_{j=1}^{n} k^j \;=\; \frac{k^{n+1} - 1}{k - 1}$$
  (By the sum formula of the previous slide.)

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Example The set of natural numbers $\mathbb{N}$ and the set of even numbers $even := \{2n \mid n \in \mathbb{N}\}$ have the same cardinality, because $f : \mathbb{N} \to even$ with $f(n) = 2n$ is a bijection

Countable Sets

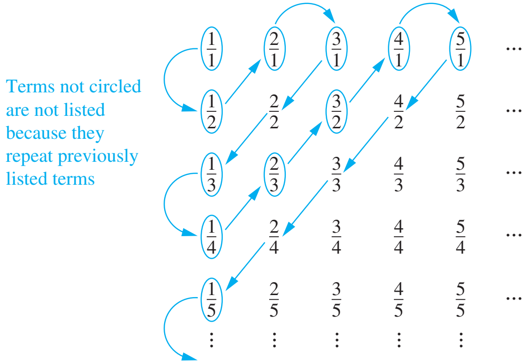
Definition

- A set S is called countably infinite, iff it has the same cardinality as the natural numbers, $|S| = |\mathbb{N}|$
- A set is called countable iff it is either finite or countably infinite
- A set that is not countable is called uncountable

The positive rational numbers are countable

Construct a bijection $f : \mathbb{N} \to \mathbb{Q}^+$:

- List fractions p/q with $q = n$ in the n^{th} row
- f traverses this list in the following order
  - ▷ For $n = 1, 2, 3, \ldots$ do visit all p/q with $p + q = n$



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- First define an (alphabetical) ordering on the symbols in Σ Show that the strings can be listed in a sequence
  - First all strings of length 0 in lexicographic order
  - Then all strings of length 1 in lexicographic order
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In particular, the set of all Java-programs is countable, since every program is just a finite string

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Proof.

(Sketch) Since S_1, S_2 are countably infinite, there must exist bijections $f_1 : \mathbb{N} \to S_1$ and $f_2 : \mathbb{N} \to S_2$. Consider the disjoint parts S_1 and $S_2 - S_1$. If $S_2 - S_1$ is finite then consider this part separately and build a bijection $f : \mathbb{N} \to S_1 \cup S_2$ by shifting f_1 by $|S_2 - S_1|$. Otherwise, construct bijections between the two parts and the even/odd natural numbers, respectively. $\square$

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Assume by contraposition that a bijection $f : \mathbb{N} \to \mathit{InfiniteStrings}$ exists. Let d_n be the n^{th} symbol of string $f(n)$. We define a string x such that the n^{th} symbol of x is $d_n + 1 \pmod 2$.Thus $\forall n \in \mathbb{N}.\ x \neq f(n)$ and f is not a surjection. Contradiction $\square$

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Similarly for the infinite decimal strings (over digits $\{0, 1, 2, \ldots, 9\}$). Just use modulo 10 instead of modulo 2

The technique used in the proof above is called diagonalization

The real numbers are uncountable

A similar diagonalization argument shows uncountability of $\mathbb{R}$

Theorem

The real numbers in the interval $(0,1) \subseteq \mathbb{R}$ *are uncountable*

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The real numbers $\mathbb{R}$ *are uncountable*

Proof.

Find a bijection between $(0,1)$ and $\mathbb{R}$. *E.g.*
$f(x) = tan(\pi x - \pi/2)$ $\square$