Functions[1]

Myrto Arapinis
School of Informatics
University of Edinburgh

September 29, 2014

[1] Slides mainly borrowed from Richard Mayr

Functions as relations

A relation is a function iff each element of its domain is related to at most one element of its codomain

Definition

Let A and B be two nonempty sets. A relation $f \subseteq A \times B$ is called a **partial function** from A to B iff

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- For all $a \in (A \setminus \mathcal{D}_f)$, we say that $f(a)$ is undefined
- $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ are equal iff $A = A'$, $B = B'$ and $\forall a \in A.\ f(a) = f'(a)$

Example

Consider the function $\sqrt{\cdot} : \mathbb{R} \to \mathbb{R}$.

- $\mathcal{D}_{\sqrt{\cdot}} = (\mathbb{R}^+ \cup \{0\})$
  Note that the domain of a function, and its domain of definition do not necessarily coincide

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  Note that the codomain of a function, and its range do not necessarily coincide
- For all $x \in \mathbb{R}^-$, f is undefined at x

Total functions

Definition

A partial function $f : A \to B$ is called a **total function**[a] iff every element in A is related to exactly one element in B, *i.e.*

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

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Example The identity function over any set A is a total function

Cardinality

Theorem

Let A and B be two finite sets. The set of all relations from A to B, denoted $Rel(A, B)$, has cardinality $2^{|B||A|}$

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$$tFun(A,B) \;\subseteq\; pFun(A,B) \;\subseteq\; Rel(A,B)$$

Injective functions

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Function composition

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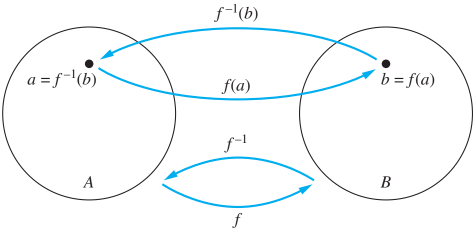
Let $f : B \to C$ and $g : A \to B$. The composition function $f \circ g$ is defined by $f \circ g : A \to C$ with $f \circ g(a) = f(g(a))$



The common notation differs between functions and relations. For functions $f \circ g$ means "first apply g, and then apply f". For relations $R_1 \circ R_2$ means "first R_1, and then R_2"

Function composition

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Corollary

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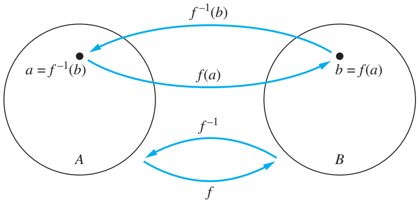
Inverse function

Definition

If $f : A \to B$ is a bijection, then the **inverse** of f, denoted f^{-1} is defined as the function $f^{-1} : B \to A$ such that $f^{-1}(b) = a$ iff $f(a) = b$



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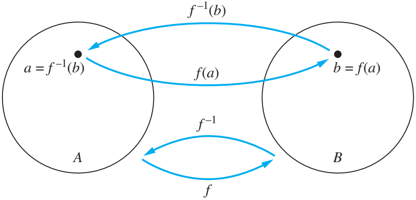

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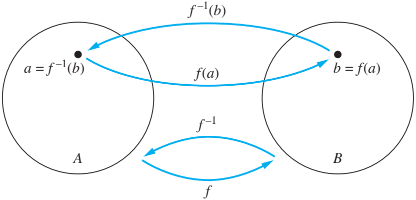

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What is the inverse of $\sqrt{}:\mathbb{R}^+ \to \mathbb{R}^+$?

What is the inverse of $\cdot + 1 : \mathbb{R} \to \mathbb{R}$?

The floor and ceiling functions

Definition

The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$

Definition

The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$

Example

$$\left\lfloor \frac{1}{2} \right\rfloor = \left\lceil -\frac{1}{2} \right\rceil = \lfloor 0 \rfloor = \lceil 0 \rceil$$

Useful properties of the floor and ceiling functions

Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

(1a) $\lfloor x \rfloor = n$ iff $n \leq x < n + 1$

(1b) $\lceil x \rceil = n$ iff $n - 1 < x \leq n$

(1c) $\lfloor x \rfloor = n$ iff $x - 1 < n \leq x$

(1c) $\lceil x \rceil = n$ iff $x \leq n < x + 1$

(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Exercise

Prove that

$$\forall x \in \mathbb{R}.\ \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

The factorial function

Definition

The factorial function $f : \mathbb{N} \to \mathbb{N}$, denoted as $f(n) = n!$ assigns to n the product of the first n positive integers

$$f(0) = 0! = 1$$

and

$$f(n) = n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$$