Relations[1]

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[1]Slides mainly borrowed from Richard Mayr

Relations

Definition

Given sets A and B, $R \subseteq A \times B$ is a binary relation from A to B, denoted $R : A \to B$

- R is a set of ordered pairs, *i.e.* $R \in \mathcal{P}(A \times B)$
- A is called the domain of R
- B is called the codomain of R
- We write aRb whenever $(a, b) \in R$
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Definition

Given sets $A_1, \ldots, A_n$,a subset $R \subseteq A_1 \times \cdots \times A_n$ is an n-ary relation

Informal examples

- Computation
- Typing
- Program equivalence
- Networks
- Databases

Examples

Empty relation -

- $\emptyset : A \to B$
- $\forall a \in A.\ \forall b \in B.\ \neg(a \emptyset b)$

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Identity relation -

- $I_A : A \to A$
- $I_A = \{(a, a) \mid a \in A\}$
- $\forall a_1, a_2 \in A.\ ((a_1 I_A a_2) \leftrightarrow (a_1 = a_2))$

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Divides relation -

- $| : \mathbb{Z}^+ \to \mathbb{Z}^+$
- $| = \{(n, m) \mid \exists k \in \mathbb{Z}^+.\ m = kn\}$
- $\forall n, m \in \mathbb{Z}^+.\ ((n \mid m) \leftrightarrow (\exists k \in \mathbb{Z}^+.\ m = kn))$

Properties of binary relations

A binary relation $R : A \to A$ is called

- reflexive iff $\forall x \in A.\ (x, x) \in R$

  Examples $\leq$, $=$, and $|$ are reflexive, but $<$ is not

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  Examples $\leq$, $=$, $<$, and $|$ are antisymmetric

- transitive iff
  $\forall x, y, z \in A.\ (((x, y) \in R \wedge (y, z) \in R) \ \to \ (x, z) \in R)$

  Examples $\leq$, $=$, $<$, and $|$ are transitive

Combining relations

Since relations are sets, they can be combined with normal set operations, *e.g.* $< \cup =$ is equal to $\leq$, and $\leq \cap \geq$ is equal to $=$. Moreover, relations can be composed.

Definition

Let $R_1 : A \to B$ and $R_2 : B \to C$.Then R_1 is composable with R_2. The composition is defined by

$$R_1 \circ R_2 \stackrel{\text{def}}{=} \{(x, z) \in A \times C \mid \exists y \in B.\ ((x, y) \in R_1 \wedge (y, z) \in R_2)\}$$

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Theorem

Relational composition is associative and has the identity relation as neutral element

- Associativity - (proof on the board)

$$\forall R : A \to B,\ S : B \to C,\ T : C \to D,\ (T \circ S) \circ R = T \circ (S \circ R)$$

- Neutral element - (proof on the board)

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Corollary

For every set A, the structure $(\mathcal{P}(A \times A), I_A, \circ)$ is a monoid

Powers of a relation

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Given a relation $R \subseteq A \times A$ on A, its powers are defined inductively by

Base step: $R^0 = I_A$

Induction step: $R^{n+1} = R^n \circ R$

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If R is a transitive relation, then its powers are contained in R itself. Moreover, the reverse implication also holds.

Theorem

A relation R on a set A is transitive iff $R^n \subseteq R$ for all $n = 1, 2, \ldots$

Equivalence relations

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Example Let Σ^* be the set of strings over alphabet Σ. Let $R \subseteq \Sigma^* \times \Sigma^*$ be a relation on strings defined as follows $R = \{(s, t) \in \Sigma^* \times \Sigma^* \mid |s| = |t|\}$. R is an equivalence relation

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Example Let $R = \{(n,m) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid n \mid m\}$. R is not an equivalence relation (proof on the board)

Congruence modulo m

Let $m > 1$ be an integer, and $R = \{(a, b) \mid a = b \pmod{m}\}$. R is an equivalence on the set of integers

Equivalence classes

Definition

Let R be an equivalence relation on a set A and $a \in A$. Let

$$[a]_R = \{s \mid (a, s) \in R\}$$

be the equivalence class of a *w.r.t.* R, *i.e.* all elements of A that are R-equivalent to a

If $b \in [a]_R$ then b is called a representative of the equivalence class. Every member of the class can be a representative

Theorem

Theorem

Let R be an equivalence on A and $a, b \in A$. The following three statements are equivalent

1. aRb
2. $[a]_R = [b]_R$
3. $[a]_R \cap [b]_R \neq \emptyset$

(proof on the board)

Partitions of a set

Definition

A partition of a set A is a collection of disjoint, nonempty subsets that have A as their union. In other words, the collection of subsets $A_i \subseteq A$ with $i \in I$ (where I is an index set) forms a partition of A iff

1. $A_i \neq \emptyset$ for all $i \in I$
2. $A_i \cap A_j = \emptyset$ for all $i \neq j \in I$
3. $\bigcup_{i \in I} A_i = A$

Theorem

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1. If R is an equivalence on A, then the equivalence classes of R form a partition of A
2. Conversely, given a partition $\{A_i \mid i \in I\}$ of A there exists an equivalence relation R that has exactly the sets A_i, iI, as its equivalence classes

(proof on the board)

Partial orders

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Example The relation $|$ is a partial order, *i.e.* $(\mathbb{Z}^+, |)$ is a poset

Example Set inclusion $\subseteq$ is partial order, *i.e.* $(2^A, \subseteq)$ is a poset

Comparability and total orders

Definition

Two elements a and b of a poset (S, R) are called comparable iff aRb or bRa holds. Otherwise they are called incomparable

Definition

If (S, R) is a poset where every two elements are comparable, then S is called a totally ordered or linearly ordered set and the relation R is called a total order or linear order

Extending orders to tuples: Standard

Let $(S, \preceq)$ be a poset and $S^n = S \times S \times \ldots \times S$ (n times)
The standard extension of the partial order to tuples in S^n is defined by

$$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \leftrightarrow \forall i \in \{1, \ldots, n\}.\ x_i \preceq y_i$$

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Exercise Prove that this defines a partial order.

Note Even if $(S, \preceq)$ is totally ordered, the extension to S^n is not necessarily a total order. Consider $(\mathbb{N}, \leq)$. Then
$(2, 1) \not\leq (1, 2) \not\leq (2, 1)$

Extending orders to tuples: Lexicographic

Let $(S, \preceq)$ be a poset and $S^n = S \times S \times \ldots \times S$ (n times). The lexicographic order on tuples in S^n is defined by

$$(x_1, \ldots, x_n) \prec_{lex} (y_1, \ldots, y_n) \leftrightarrow \exists i \in \{1, \ldots, n\}. \forall k < i.\, x_k = y_k \wedge x_i \prec y_i$$

$(x_1, \ldots, x_n) \preceq_{lex} (y_1, \ldots, y_n)$ iff $(x_1, \ldots, x_n) \prec_{lex} (y_1, \ldots, y_n)$ or $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$

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Lemma

If $(S, \preceq)$ is totally ordered then $(S^n, \preceq_{lex})$ is totally ordered