Predicate logic[1]

Myrto Arapinis
School of Informatics
University of Edinburgh

September 18, 2014

[1] Slides mainly borrowed from Richard Mayr

Propositional logic is not enough

In proposition logic, from:

- All men are mortal
- Socrates in a man

we cannot derive that:

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We need a language to talk about objects, their properties and their relations

Predicate logic

Extends propositional logic by the following new features:

- Variables: x, y ,z, ...
- Predicates (*i.e.* propositional functions): $P(x)$, $Q(x)$, $R(y)$, $M(x, y)$, ...
- Quantifiers: $\forall$, $\exists$

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Propositional functions are a generalization of propositions"

- Can contain variables and predicates, *e.g.* $P(x)$
- Variables stand for (and can be replaced by) elements from their domain
- The truth value of a predicate depends on the values of its variables

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Example: Let $P(x)$ denote "$x > 5$" and $\mathcal{U} = \mathbb{Z}$. Then

- P(8) is true
- P(5) is false

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- The quantifiers are said to bind the variable x in these expressions. Variables in the scope of some quantifier are called bound variables. All other variables in the expression are called free variables
- A propositional function that does not contain any free variables is a proposition and has a truth value

Nested quantifiers

Complex meanings require nested quantifiers

Example: "Every real number has an inverse *w.r.t.* addition"
Let the domain $\mathcal{U} = \mathbb{R}$. Then the property is expressed by

$$\forall x.\ \exists y.\ (x + y = 0)$$

Example: "Every real number except zero has a multiplicative inverse.
Let the domain $\mathcal{U} = \mathbb{R}$. Then the property is expressed by

$$\forall x.\ (x \neq 0 \ \rightarrow \ \exists y.\ (x \times y = 1)$$

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- Example: $\forall x.\ \forall y.\ P(x, y) \equiv \forall y.\ \forall x.\ P(x, y)$

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- Example: $\forall x.\ \forall y.\ P(x, y) \equiv \forall y.\ \forall x.\ P(x, y)$
- Example: $\forall x.\ \exists y.\ P(x, y) \not\equiv \exists y.\ \forall x.\ P(x, y)$

Quantifiers as Conjunctions/Disjunctions

- If the domain is finite then universal/existential quantifiers can be expressed by conjunctions/disjunctions.
  Example: If $\mathcal{U}$ consists of the integers 1,2, and 3, then
  - $\forall x.\ P(x) \equiv P(1) \wedge P(2) \wedge P(3)$
  - $\exists x.\ P(x) \equiv P(1) \vee P(2) \vee P(3)$
- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long

De Morgan's law for quantifiers

The rules for negating quantifiers are:

- $\neg(\forall x.\ P(x)) \equiv \exists x.\ \neg P(x)$
- $\neg(\exists x.\ P(x)) \equiv \forall x.\ \neg P(x)$

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An assertion in predicate calculus is valid iff it is true

- for all domains
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Example: $\forall x.\ P(x) \vee (\neg P(x))$

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Example: $\forall x.\ P(x) \wedge (\neg P(x))$ is unsatisfiable

If x is irrational then so is $\sqrt{x}$

Let Irrational(x) denote the propositional function "x is irrational", and Rational(x) $= \neg$Irrational(x)

Proposition

$\forall x \in \mathbb{R}^+.\ \text{Irrational}(x) \ \rightarrow\ \text{Irrational}(\sqrt{x})$

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Proof Let x be positive real number. We will show the contrapositive,*i.e.* $\forall x \in \mathbb{R}^+.\ \text{Rational}(\sqrt{x}) \ \rightarrow\ \text{Rational}(x)$. In other words we prove that if $\sqrt{x}$ is rational then so is x. Assume that $\sqrt{x}$ is a rational number. Then, by definition, there must exists two natural numbers m and n such that $\sqrt{x} = m/n$. But then $x = m^2/n^2$ and, since m^2 and n^2 are natural numbers, which by definition implies that x is a rational number as required. $\square$

n is even iff n^2 is even.

Let $\text{Even}(x)$ denote the propositional function "x is even, and $\text{Odd}(x) = \neg\text{Even}(x)$

Proposition

$\forall n \in \mathbb{Z}.\ \text{Even}(n) \ \leftrightarrow\ \text{Even}(n^2)$

n is even iff n^2 is even.

Let $\text{Even}(x)$ denote the propositional function "x is even, and $\text{Odd}(x) = \neg\text{Even}(x)$

Proposition

$\forall n \in \mathbb{Z}.\ \text{Even}(n) \ \leftrightarrow\ \text{Even}(n^2)$

Proof Let $n \in \mathbb{Z}$

($\rightarrow$) Let's assume that n is even, *i.e.* there exists an integer k such that $n = 2k$. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, and thus for $\ell = 2k^2$, $n^2 = 2\ell$ and so is even.

($\rightarrow$) We will show the contrapositive, *i.e.* $\text{Odd}(n) \ \rightarrow\ \text{Odd}(n^2)$. Let's assume that n is odd, *i.e.* there exists an integer k such that $n = 2k + 1$. Then $n^2 = (2k+1)^2 = 2(2k^2 + 2k) + 1$, and thus for $\ell = 2k^2 + k$, $n^2 = 2\ell + 1$ and so is odd. $\square$

$4k = i^2 - j^2$

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Proof Let $k \in \mathbb{N}$. Let $i = k + 1$ and $j = k + 1$.
$i^2 - j^2 = (k+1)^2 - (k-1)^2 = k^2 + 2k + 1 - k^2 + 2k - 1 = 4k.\square$

Either $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

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Proof Let $n \in \mathbb{Z}$. n is either even or odd. We consider each case separately.
(1) Assume n is even. Then there exists m such that $n = 2m$. But then $n^2 = 4m^2 \equiv 0 \pmod 4$
(1) Assume n is odd. Then there exists m such that $n = 2m + 1$. But then $n^2 = 4m^2 + 4m + 1 = 4(m^2 + m) + 1 \equiv 1 \pmod 4$ $\square$