DISCRETE MATHEMATICS AND MATHEMATICAL REASONING

# Mock Exam SOLUTIONS, Fall 2012

## INSTRUCTIONS TO CANDIDATES

- **Answer ALL FOUR questions.**
  **Each of the four questions has the same total number of marks: 25.**

  **You have 2 hours to complete the (mock) exam.**

**Note:** this is **NOT** an actual exam. It is only intended to give you a general impression of the number and size of the questions, and rough level of difficulty of the questions, that will be on the actual exam. Many specific topics that can and will be covered on the actual exam are necessarily not covered in this mock exam, because there are simply too many topics covered in the full DMMR course to cover all of them in any single 2-hour exam.

So, do not assume that by doing well on this mock exam you are likely to do well on the actual exam. You need to study all of the topics covered in the DMMR course to prepare well for the exam. (Follow the study guide on the DMMR course web page, to see what topics were covered.)

1. (a) Let $a$ and $b$ be distinct elements. For each of the following sets, prove or disprove that it is the powerset of another set. [8 marks]

   i. $\emptyset$
   ii. $\{\emptyset, \{a\}\}$
   iii. $\{\emptyset, \{a\}, \{\emptyset, a\}\}$
   iv. $\{\emptyset, a, b, \{a, b\}\}$

   **Solution:**

   i. $\emptyset$ is not the powerset of any set. For any set $S$ we have $|2^S| = 2^{|S|} \geq 1 > 0 = |\emptyset|$.
   ii. $\{\emptyset, \{a\}\}$ is the powerset of $\{a\}$.
   iii. $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ is not the powerset of any set. For any set $S$ we have that either $|2^S| = 2^{|S|} = 1$ or $|2^S| = 2^{|S|}$ is even. Since $|\{\emptyset, \{a\}, \{\emptyset, a\}\}| = 3$ it is not the powerset of any set.
   iv. $\{\emptyset, a, b, \{a, b\}\}$ is not the powerset of any set. Assume the opposite the derive a contradiction. If $\{\emptyset, a, b, \{a, b\}\} = 2^S$ for some set $S$ then $\{a, b\} \subseteq S$. Therefore $\{a\} \subseteq S$ and thus $\{a\} \in 2^S = \{\emptyset, a, b, \{a, b\}\}$. Contradiction.

   (b) Give a big-$\mathcal{O}$ estimate for the following function. Use a simple function $g$ of the lowest possible order. [9 marks]

$$f(n) = (2^n + n^2)(n^3 + 3^n)$$

   **Solution:** We have that $(2^n + n^2)$ is $\mathcal{O}(2^n)$, since $2^n \geq n^2$ for $n \geq 4$. Also $(n^3 + 3^n)$ is $\mathcal{O}(3^n)$, since $3^n \geq n^3$ for $n \geq 6$. Thus $f(n)$ is $\mathcal{O}(2^n)\mathcal{O}(3^n)$ and thus $\mathcal{O}(6^n)$.

   (c) Prove or disprove the following property. If $ac \equiv bc \pmod{m}$, where $a, b, c, m$ are integers and $m \geq 2$ then $a \equiv b \pmod{m}$. [8 marks]

   **Solution:** This property is false. Consider the counterexample of $m = c = 2$ and $a = 0$ and $b = 1$.

2. (a) How many different onto functions (i.e., surjective functions), $f : A \to B$, are there from the set $A = \{1, \ldots, 4\}$ to the set $B = \{a, b, c\}$? [5 marks]

**Solution: (Apologies: this question was actually more difficult than intended, and was put on the mock exam by mistake.)** Recall that a function, $f : A \to B$ is called *onto* or *surjective* iff for all $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

Since $|A| = 4$ and $|B| = 3$, there are clearly $3^4$ arbitrary functions $f : A \to B$. We have to subtract from this those functions that are not onto. We can equate each function $f : A \to B$ with a string of length 4 over the alphabet $B = \{a, b, c\}$.

We have to exclude those strings of length 4 where not every letter in $B$ occurs. Consider those strings where only $\{a, b\}$ occur. There are $2^4$ of those. There are likewise $2^4$ strings where only letters in $\{b, c\}$ occur, and likewise $2^4$ strings where only letters in $\{a, c\}$ occur.

So, we can consider $3^4 - 3 \cdot 2^4$ as a partial calculation of the number we are after. However, this is not the correct number, because in our substraction we account for each string were only a single letter occurs twice. There are three such strings: $\{aaaa, bbbb, cccc\}$, so we have to add back 3.

This completes the calculation, and leads to the overall answer:

$$3^4 - 3 \cdot 2^4 + 3 = 3 * (27 - 16 + 1) = 36$$

(This is an application of the general inclusion-exclusion principle, which we did not cover in class, so this was an inappropriately difficult question for the mock exam. Apologies. It will not happen on the actual exam.)

(b) How many different strings can be formed by reordering the letters of the word $CHATTANOOGA$? (Show your calculation.) [5 marks]

**Solution:** This is a straightforward application of the formula for counting permutations with indistinguishable objects.

Specifically, in this case there are 11 letters in the string $CHATTAANOOGA$, the three copies of $A$ are indistinuishable, as are the 2 copies of $T$ and the two copies of $O$. The remaining letters are all distinct.

Thus, the number of reorderings of the letter in this word is:

$$\frac{11!}{3!2!2!}$$

(You are not expected to calculate the exact number this represents. In this case the number happens to be 1663200. )

(c) A football team has 21 players, 3 of whom are "goalkeepers", 6 of whom are "defenders", 8 of whom are "midfielders", and 4 of whom are "strikers".

(Note that each of the 21 players belongs to exactly one of these 4 categories.)

In each match, exactly 11 players must start the match. Of these 11, exactly 1 must be a goalkeeper, exactly 4 must be defenders, either 4 or 5 must be midfielders, and either 2 or 1 must be strikers.

The manager has to choose which 11 players, out of the 21, start each match, subject to the above constraints, and only the above constraints.

How many different possible choices for the set of starting 11 players does the manager have?

(Show your calculations.)

[8 marks]

**Solution:** We must choose one goalkeeper, and there are 3 to choose from. So there are $\binom{3}{1} = 3$ choices for goalkeeper.

We must choose 4 defenders. There are 6 to choose from. Thus $\binom{6}{4} = 15$ choices for defenders. For the remaining players, we either choose to have 5 midfielders and 1 striker, or 4 midfielders and 2 strikers. Note that these are mutually exclusive possibilities. In the first case, case we have $\binom{8}{5} = 56$ choices for midfield and $\binom{4}{1} = 4$ choices for striker. In the second case we have $\binom{8}{4}$ choices for midfield and $\binom{4}{2}$ choices for striker, respectively.
So, overall, the number of different choices is:

$$\binom{3}{1} \cdot \binom{6}{4} \cdot (\binom{8}{5}\binom{4}{1} + \binom{8}{4}\binom{4}{2})$$

(You are not expected to calculate the exact number this represents. In this case the number happens to be 21420. )

(d) For any simple undirected graph $G = (V, E)$, its *complement graph*, $G' = (V, E')$, is defined as follows: $G$ and $G'$ have the same set of vertices, $V$, and the edges $E'$ of of $G'$ are defined as follows: for every pair of vertices $u, v \in V$, such that $u \neq v$, $\{u, v\} \in E'$ if and only if $\{u, v\} \notin E$.
Prove that two simple undirected graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if and only if their respective *complement graphs*, $G'_1$ and $G'_2$, are isomorphic. [7 marks]

**Solution:** By definition, $G_1$ and $G_2$ are isomorphic if and only if the following holds: there exists a bijection $f : V_1 \to V_2$ such that for all $u, v \in V_1$, $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$.
Note that this obviously implies that for all $u, v \in V_1$, $\{u, v\} \notin E_1$ if and only if $\{f(u), f(v)\} \notin E_2$. But this is equivalent to the condition that the complement graphs $G'_1 = (V_1, E'_1)$ and $G'_2 = (V_2, E'_2)$ are isomorphic, because by definition $\{u, v\} \notin E_i$ if and only if $\{u, v\} \in E'_i$.

3. (a) Each of the following three conditions defines a relation $R$ on the set of all integers. In each case, prove or disprove that the relation $R$ is reflexive/symmetric/antisymmetric/transitive. Let $(x, y) \in R$ if and only if [12 marks]

   i. $x = y + 1$ or $x = y - 1$
   **Solution:** Not reflexive, since $(0, 0) \notin R$. If $(x, y) \in R$ then $x = y+1$ or $x = y-1$. Therefore $y = x - 1$ or $y = x + 1$. Thus $(y, x) \in R$ and $R$ is symmetric. $R$ is not antisymmetric, since $(1, 0) \in R$ and $(0, 1) \in R$. $R$ is not transitive, since $(0, 1) \in R$ and $(1, 2) \in R$, but $(0, 2) \notin R$.

   ii. $xy \geq 1$
   **Solution:** $R$ not reflexive since $(0, 0) \notin R$. $R$ symmetric, since $xy = yx$. $R$ not antisymmetric, since $(1, 2) \in R$ and $(2, 1) \in R$. $R$ is transitive. Let $(x, y) \in R$ and $(y, z) \in R$. Then $xy \geq 1$ and $yz \geq 1$. Therefore $x, y, z \neq 0$. There are two cases: Either $x$ and $y$ are both positive and $\geq 1$ or they are both negative and $\leq -1$. (This follows from $xy \geq 1$ and the fact that $x, y$ are integers.) In the first case we must have $z \geq 1$, because $y \geq 1$, $yz \geq 1$ and $z$ is an integer. In the second case we must have $z \leq -1$, because $y \leq -1$, $yz \geq 1$ and $z$ is an integer. In both cases we obtain $xz \geq 1$ and thus $(x, z) \in R$.

   iii. $x = y^2$
   **Solution:** $R$ is not reflexive, since $(2, 2) \notin R$. $R$ is not symmetric, since $(4, 2) \in R$ and $(2, 4) \notin R$. $R$ is antisymmetric. If $(x, y) \in R$ and $(y, x) \in R$ then $x = y^2$ and $y = x^2$. Therefore $x = x^4$ and thus either $x = 0$ or $x = 1$. In both cases the condition $y = x^2$ yields $y = x$. $R$ is not transitive, since $(16, 4) \in R$ and $(4, 2) \in R$, but $(16, 2) \notin R$.

(b) Prove or disprove the following property. If $A$ is an uncountable set and $B$ is a countable set then $A - B$ is uncountable. [13 marks]

**Solution:** First we show that the disjoint union of a countably infinite set $X$ and a finite set $Y$ is countably infinite. There exists a bijection $f : X \to \mathbb{N}$, since $X$ is countably infinite. Let $Y = \{y_0, \ldots, y_{k-1}\}$, since $Y$ is finite. Then $g : X \cup Y \to \mathbb{N}$, defined by $g(y_i) = i$ and $g(x) = k + f(x)$ for $x \in X$, is a bijection.

Now we prove the main property. We assume the opposite, i.e., that $A - B$ is countable, and derive a contradiction. There are three cases:

   i. If both $B$ and $A - B$ are finite then $A = A - B \cup B$ is finite. Contradiction.

   ii. If either $B$ or $A - B$ is countably infinite and the other is finite, then $A = A - B \cup B$ is countably infinite by the property shown above. Contradiction.

   iii. If both $B$ and $A - B$ are countably infinite then there exist bijections $f : B \to \mathbb{N}$ and $g : A - B \to \mathbb{N}$. Then the function $h : A \to \mathbb{N}$, defined by $h(x) = 2f(x)$ if $x \in B$ and $h(x) = 2g(x) + 1$ if $x \in A - B$ is a bijection. Thus $A$ is countably infinite. Contradiction.

4. (a) Suppose that 0.2% of the population of the UK is known to have a particular genetic trait, which can be detected using a blood test.

   Suppose that the NHS can sample individuals from the population uniformly at random, and conduct the blood test on them.

   What is the expected number of random samples of individuals the NHS would need to carry out before they find one person who has this genetic trait. [5 marks]

   **Solutions:** From the description, the probability of detecting the genetic trait in each trial is $0.2\% = 2/1000$. We assume the different samples are independent (which seems a reasonable assumption in this case). Then number of samples needed before we find the first person who has this genertic trait is a geometrically distributed random variable, with parameter $p = 2/1000$. Thus, from the known fact that the expected value of a geometrically distributed random variable with parameter $p$ is $1/p$, the expected number of trials until the first success is: $(1/p) = 1000/2 = 500$.

   (b) Suppose $A$ and $B$ are events from a finite sample space $\Omega$. Suppose the event $C$ is defined by $C = \{s \in \Omega \mid s \notin B\}$.

   Let $P : \Omega \to [0, 1]$ be a probability distribution on $\Omega$, and suppose that we know that $P(A) = 4/5$, $P(C) = 2/3$, and $P(A \mid B) = 3/5$.

   Compute $P(B \mid A)$. (Show your calculation.) [5 marks]

   **Solutions:** Note that $C = \Omega - B$. Thus $2/3 = P(C) = 1 - P(B)$. Thus $P(B) = 1/3$. By the definition of conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \mid B)P(B)}{P(A)} = \frac{(3/5)(1/3)}{4/5} = \frac{1}{4}$$

   (c) Recall that a simple undirected graph, $G = (V, E)$, is called *bipartite* it the vertices $V$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge $\{u, v\} \in E$ connect one vertex $u$ in $V_1$ to one vertex $v$ in $V_2$.

   Prove that if a simple undirected graph is bipartite then it does not contain a simple circuit of odd length (the length of a circuit is the number of edges on it). [7 marks]

   **Solution:** Here is a **Proof:** suppose the graph contains a simple circuit of odd length, and let $v_1 v_2 \ldots v_{2k+1} v_1$ be the sequence of vertices on this circuit. Suppose, without loss of generality, that $v_1 \in V_1$. Then $v_2$ must be in $V_2$, and $v_3$ must be in $V_1$, and in general, by induction on $i$, $v_{2i}$ must be in $V_2$ whereas $v_{2i+1}$ must be in $V_1$, for all $i = 1, \ldots, k$. But this is impossible, since then both $v_{2k+1}$ and $v_1$ are in $V_1$, but there is an edge between them, so $V_1$ and $V_2$ can not form a bipartition of the graph as assumed, which is a contradiction.

   (d) Recall that a *full m-ary tree* is a rooted tree where every internal vertex has exactly $m$ children.

   Does there exist a full 3-ary tree with exactly 11 vertices?

   If yes, give an example of such a full 3-ary tree. If not, prove why not. [8 marks]

   **Solutions:** There is no full 3-ary tree with 11 vertices. This is because the number of nodes $n$ in a full $m$-ary tree with $i$ internal nodes is $n = mi + 1$. (This is because every node is the child of one internal node, except for the root, and there must be $m \cdot i$ children of internal nodes in total. So there are $mi + 1$ nodes in total.)

   Thus, in order for there to exist a full 3-ary tree with 11 nodes, it must be the case that $11 = 3 \cdot i + 1$, for some positive integer $i$. But this is impossible since this equation can be rearanged to yield that $i = 10/3$, which is not an integer. So, no such full 3-ary tree exists.