UNIVERSITY OF EDINBURGH

COLLEGE OF SCIENCE AND ENGINEERING

SCHOOL OF INFORMATICS

DISCRETE MATHEMATICS AND MATHEMATICAL REASONING

Saturday 1 $^{\underline{\text{st}}}$ April 2017

00:00 to 00:00

INSTRUCTIONS TO CANDIDATES

1. **Answer all five questions in Part A, and two out of three questions in Part B. Each question in Part A is worth 10% of the total exam mark; each question in Part B is worth 25%.**
2. **Use a single script book for all questions.**
3. **CALCULATORS MAY NOT BE USED FOR THIS EXAM.**
4. **This is an open book exam. Notes and the course textbook (K. Rosen's "Discrete Mathematics and its applications") are allowed in the exam.**

THIS EXAMINATION WILL BE MARKED ANONYMOUSLY

PART A

ANSWER ALL QUESTIONS IN PART A

1. (a) Recall that two sets have the same cardinality if there is a bijection between them and that $\mathbb{Z}$ is the set of all integers. Give an example of a bijection $f : \mathbb{Z} \to \mathbb{Z}$ which is different from the identity function. [*3 marks*]

   (b) For the following sets A prove that A has the same cardinality as the positive integers $\mathbb{Z}^+$

      i. $A = \{x \in \mathbb{Z}^+ \mid \exists y \in \mathbb{Z}\ x = y^2\}$ [*3 marks*]

      ii. $A = \mathbb{Z}$ [*4 marks*]

2. (a) Assume n is a positive integer with $n \geq 1$. Prove by induction that [*7 marks*]

$$\sum_{k=1}^{n}(a + (k-1)r) = \frac{n}{2}(2a + (n-1)r)$$

   (b) Recall $\gcd(n, m)$ for positive integers n and m is the greatest common divisor of n and m. Using Euclid's algorithm, compute $\gcd(89, 55)$. [*3 marks*]

3. (a) How many distinct functions $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4\}$ are there, from the set $\{1, 2, 3, 4, 5, 6\}$ to the set $\{1, 2, 3, 4\}$, such that for all $i \in \{1, 2, 3\}$, $f(7 - i) = 5 - f(i)$. [*5 marks*]

   (b) How many distinct functions $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{1, 2, 3, 4, 5, 6, 7\}$ are there, from the set $\{1, 2, 3, 4, 5, 6, 7\}$ to itself, such that there does not exist any $i \in \{1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) = i$. [*5 marks*]

4. Prove that for all positive integers n and d such that $n \geq d \geq 1$, the following inequality holds:

$$\sum_{i=0}^{d}\binom{n}{i} \leq (n+1)^d$$

[*10 marks*]

5. Suppose that a particular gene in chimpanzees has been identified, such that if a chimpanzee has disease A, then with probability 3/5 the chimp has this gene, whereas if it does not have disease A, then with probability 1/10 it has this gene. Suppose that the probability that a random chimp has disease A is 1/6. What is the probability that a random chimp has disease A, given that it has that gene? [*10 marks*]

PART B

ANSWER TWO QUESTIONS FROM PART B

6. This question concerns $\mathcal{P}(A)$ which is the powerset of A, the set of all subsets of A. Determine which of the following statements are true and which are false. Prove each statement that is true and give a counterexample for each statement that is false.

   (a) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ *[3 marks]*

   (b) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ *[3 marks]*

   (c) $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$ *[3 marks]*

   (d) $\mathcal{P}(A) \times \mathcal{P}(B) \subseteq \mathcal{P}(A \times B)$ *[3 marks]*

   (e) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ *[3 marks]*

   (f) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ *[4 marks]*

   (g) $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$ *[3 marks]*

   (h) $\mathcal{P}(A \times B) \subseteq \mathcal{P}(A) \times \mathcal{P}(B)$ *[3 marks]*

7. Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ be arbitrary integers and consider the system of equations

$$\begin{aligned}
x &\equiv a_1 \pmod{m_1} \\
x &\equiv a_2 \pmod{m_2} \\
&\vdots \\
x &\equiv a_n \pmod{m_n}
\end{aligned}$$

Let $m = m_1 m_2 \cdots m_n$

(a) Prove that $x = a_1 M_1 y_1 + \ldots + a_n M_n y_n$ is a solution to the equation system, where $M_i = m/m_i$ and y_i is an inverse of M_i. Give full details of the steps of your proof. *[10 marks]*

(b) Find a solution to the following system, explaining your calculations. *[10 marks]*

$$\begin{aligned}
x &\equiv 1 \pmod{2} \\
x &\equiv 2 \pmod{3} \\
x &\equiv 4 \pmod{5} \\
x &\equiv 6 \pmod{7}
\end{aligned}$$

(c) Prove that if z is also a solution to the system of equations then z is equivalent to x modulo m, $z \equiv x \pmod{m}$. *[5 marks]*

8. Consider a simple undirected graph $G = (V, E)$, with $n = |V| \geq 3$ vertices and m edges. Suppose that G is *triangle-free*, meaning that there do not exist three distinct vertices $v_1, v_2, v_3 \in V$, such that $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\} \subseteq E$. For a vertex $u \in V$, let $d(u)$ denote the degree of u. Prove that for such a triangle-free graph G, the following all hold:

   (a) there exists a vertex $u \in V$ with $d(u) \leq \frac{n}{2}$. *[6 marks]*

   (b) $m \leq \frac{n^2}{4}$. (Hint: use induction on $n \geq 3$.) *[14 marks]*

   (c) If $n > 5$, then there exists three distinct vertices $w_1, w_2, w_3 \in V$, no two of which have an edge between them. *[5 marks]*