1. Use strong induction to show that every positive integer $n$ can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. 

[Hint: For the inductive step at stage $k + 1$, separately consider the case where $k + 1$ is even and where it is odd.]

Before beginning your proof, state the property (the one you are asked to prove for every integer $n$) in completely formal notation with all quantifiers.

**Solution:**

The sentence we are required to prove can be stated formally as follows:

$$
\forall n \in \mathbb{Z}^+ \exists a_1, a_2, a_3, \ldots, a_m \in \mathbb{N} \\
[(\forall i, j \in \{1, \ldots, m\} \ i \neq j \rightarrow a_i \neq a_j) \land n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m}] 
$$

or more concisely as follows:

$$
\forall n \in \mathbb{Z}^+ \exists S \subseteq \mathbb{N} \left(n = \sum_{a \in S} 2^a\right)
$$

**Base case:** The sum with a single element $2^0$ equals 1.

**Inductive Hypothesis:** We assume that every $l$, with $l \leq k$, is the sum of distinct powers of two and then prove it for $k + 1$ by splitting in cases where $k + 1$ is even and when it is odd.

**Inductive Step:** Case 1: If $k + 1$ is even then $(k+1)/2$ is an integer and $(k+1)/2 \leq k$. Using the inductive hypothesis we can write $(k+1)/2$ as $2^{a_1} + 2^{a_2} + \cdots + 2^{a_m}$ where all $a_i$’s are distinct. Then,

$$
k + 1 = 2(2^{a_1} + 2^{a_2} + \cdots + 2^{a_m}) = 2 \cdot 2^{a_1} + 2 \cdot 2^{a_2} + \cdots + 2 \cdot 2^{a_m} = 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_m+1}
$$

These are clearly distinct powers of two, thus we have proved what we wanted.

Case 2: If $k + 1$ is odd we apply the inductive hypothesis to $k$ to get $k = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m}$. However, we know that $a_i \neq 0$ for every $i$ with $1 \leq i \leq m$ because otherwise exactly one element of the sum would be $2^0 = 1$ and the rest would be even numbers, and thus $k$ would be odd. Therefore,

$$
k + 1 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m} + 1 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m} + 2^0
$$

and $\{2^{a_1}, 2^{a_2}, \ldots, 2^{a_m}, 2^0\}$ are all distinct. \qed
2. What is wrong with this “proof”?

“Theorem” For every positive integer \(n\), if \(x\) and \(y\) are positive integers with \(\max(x, y) = n\), then \(x = y\).

**Base case:** Suppose that \(n = 1\). If \(\max(x, y) = 1\) and \(x\) and \(y\) are positive integers, we have \(x = 1\) and \(y = 1\).

**Inductive Hypothesis:** Let \(k\) be a positive integer. Assume that whenever \(\max(x, y) = k\) and \(x\) and \(y\) are positive integers, then \(x = y\). Now let \(\max(x, y) = k + 1\), where \(x\) and \(y\) are positive integers.

**Inductive Step:** Then \(\max(x - 1, y - 1) = k\), so by the inductive hypothesis, \(x - 1 = y - 1\). It follows that \(x = y\), completing the inductive step.

**Solution:**

The result is clearly false, so the proof must be wrong. The base case proof is correct, so the problem has to be in the inductive step. The inductive hypothesis is stated correctly and it is true that if \(\max(x, y) = k + 1\) then \(\max(x - 1, y - 1) = k\), so the problem must be in applying the inductive hypothesis. Analysing the inductive hypothesis we see that it requires the numbers to be positive integers to conclude that they are equal. However, it is applied to the predecessors \(x - 1\) and \(y - 1\) of two positive integers, which are not necessarily positive. By incrementing the size of \(k\) starting from its value in the base case (1) we can find the place where the chain of dominoes first breaks. For \(k = 2\) take \(x = 2\) and \(y = 1\). Then, \(\max(x, y) = 2\). However, \(x - 1 = 1\) and \(y - 1 = 0\). These numbers are not the values of \(x\) and \(y\) that we used for the base case; and if 0 had been allowed we would not have been able to prove the base case. Thus, the inductive chain breaks after the first domino.

□

3. Let \(n \geq 0\) be an integer. Prove by induction:

(a) 8 divides \(3^{2n+2} + 7\)

(b) 64 divides \(3^{2n+2} + 56n + 55\)

**Solution:**

(a) We prove this as induction over \(n\):

**Base case:** For \(n = 0\) we get \(3^2 + 7 = 9 + 7 = 16\). 8 divides 16 since 16 = 2 \cdot 8

**Induction Hypothesis:** Assume 8 divides \(3^{2n+2} + 7\).

**Induction Step:** with \((n+1)\) we get

\[
3^{2(n+1)+2} + 7 \\
= 3^{2n+2+2} + 7 \\
= 3^{2n+2} \cdot 9 + 7 \\
= 3^{2n+2} \cdot 8 + 3^{2n+2} + 7 \\
\]

By the IH we know that \(3^{2n+2} + 7\) is presentable as \(c \cdot 8\). Therefore we get \(3^{2(n+1)+2} + 7 = (3^{2n+2} + c) \cdot 8\). Since \((3^{2n+2} + c) \in \mathbb{Z}\) this means 8 divides \(3^{2(n+1)+2} + 7\) by definition.

By the induction principle 8 divides \((3^{2n+2} + 7)\) for every \(n \geq 0\)

(b) Proof by induction over \(n\):

**Base case:** For \(n = 0\) we get \(3^2 + 55 = 9 + 55 = 64\). 64 divides 64 since 64 \cdot 1 = 64.

**Induction Hypothesis:** Assume 64 divides \(3^{2n+2} + 56n + 55\) for some \(n \geq 0\)
5. A continued fraction is either an integer or of the form \(n + (1/F)\) where \(F\) is a continued fraction. For example, \(7/9 = 0 + 1/(9/7)\), \(9/7 = 1 + 1/(7/2)\), \(7/2 = 3 + 1/2\); so, \(7/9 = 0 + 1/(1 + 1/(3 + 1/2))\). Similarly, \(17/14 = 1 + 1/(4 + 1/(1 + 1/2))\). What you have to prove is that every rational can be expressed as a continued fraction. Let \(P(k)\) be “any rational with denominator \(k\) can be expressed as a continued fraction”. Prove by strong induction \(\forall x \in \mathbb{Z^+}(P(x))\).

In your proof you can use the division algorithm: if \(a\) is an integer and \(d\) a positive integer then there are unique integers \(q\) and \(r\), with \(0 \leq r < d\) such that \(a = dq + r\).

**Solution:**

**Base case:** Show \(P(1)\). Consider any rational with denominator 1. Assume it is \(n/1\). Since \(n\) is a continued fraction and \(n = n/1\), \(P(1)\) holds.

**Induction step:** Assume that for some \(d \in \mathbb{Z^+}\) with \(d > 1\) that for any \(k \in \mathbb{Z^+}, 1 \leq k < d, P(k)\) is true; so, any rational with denominator \(k\) can be expressed as a continued fraction. We show \(P(d)\). Consider a rational \(n/d\) with denominator \(d\). Use the division algorithm and write \(n = dq + r\) where \(1 \leq r < d\). We split into two cases.

(a) \(r = 0\). Then \(n = dq\) so \(n/d = q\) and, therefore, \(q\) is a continued fraction for \(n/d\).
(b) $r \neq 0$. Since $n = dq + r$, it follows that $n/d = q + r/d = q + 1/(d/r)$. As $1 \leq r < d$ by the inductive hypothesis there is a continued fraction $F$ for $d/r$. Therefore, $q + 1/F$ is a continued fraction for $n/d$.

□

Solutions (to the last question on the sheet) must be handed in on paper at the ITO by Wednesday, 19 October, 4:00pm. Please post it into the grey metal box on the wall outside the ITO.