DMMR Tutorial sheet 3

Relations (part 2), Recurrences, Cardinality

October 3rd, 2019

1. Many program analysis methods rely on call graphs. A call graph is a binary relation R on function (or method) names in a program. A pair $(f, g) \in R$ when the body of function (method) f in the program calls the function (method) g. For example, consider the following abstracted code for a function (method) f where we have taken out all the parameters.

```
function f() {
  g();
  h()
}
```

This means that in the body of f both g and h are called. So, the pairs $(f, g) \in R$ and $(f, h) \in R$. In turn the functions (methods) g, h may call other functions (methods).

The transitive closure of relation R, written R^+, is the following binary relation: $(f, g) \in R^+$ iff $(f, g) \in R$ or there is $n \geq 1$ such that $(f, f_1) \in R, (f_1, f_2) \in R, \ldots, (f_n, g) \in R$. That is, there is a path from f to g of consecutive pairs from R.

The symmetric closure of relation R, written R^s, is the following binary relation: $(f, g) \in R^s$ iff $(f, g) \in R$ or $(g, f) \in R$.

(a) Prove that R^+ is transitive.

Solution:
For transitivity, we need to show if $(f, g) \in R^+$ and $(g, h) \in R^+$ then $(f, h) \in R^+$. Since $(f, g) \in R^+$ we know that $(f, g) \in R$ or there exists a sequence of pairs $(f, f_1), (f_1, f_2), ..., (f_n, g)$ each in R. Furthermore, as $(g, h) \in R^+$ we know there $(g, h) \in R$ or there is a path $(g, g_1), (g_1, g_2), ..., (g_m, h)$ with pairs in R. So, there is a path which is the witness for $(f, h) \in R^+$: either $(f, g), (g, h)$ or $(f, g), (g, g_1), (g_1, g_2), ..., (g_m, h)$ or $(f, f_1), (f_1, f_2), ..., (f_n, g), (g, h)$ or $(f, f_1), (f_1, f_2), ..., (f_n, g), (g, g_1), (g_1, g_2), ..., (g_m, h)$. $\qquad\square$

(b) Explain what information the relations $(R^s)^+$ and $(R^+)^s$ contain about the function (method) calls in the program.

Solution:
The relation $(R^s)^+$ contains all pairs (f, g) of functions in R^s or where there is a path from f to g, where each consecutive pair is in R or its converse is in R. The only functions that are not in $(R^s)^+$ are those pairs, which are from completely separate parts of the program. Therefore if a pair (f, g) is not contained in this relation, the program can be split into two programs, one of which contains f, the other contains g.

The relation $(R^+)^s$ contains all pairs in R^+ and their converses. Therefore each pair (f, g) is in this relation, iff the two functions f and g could be called in the same execution of the program. This does not mean they can be separated, as two functions which are never called in the same execution together might still call the same sub-function. $\qquad\square$

(c) Decide which of the two relations $(R^s)^+$ and $(R^+)^s$ subsumes the other, give a formal proof of your claim and show an example relation R and a pair of functions which is contained in only one of them.

Solution:
The example code above is such that $(R^s)^+$ is strictly larger than $(R^+)^s$: the pair (g, h) is only contained in the first: $(f, g), (f, h) \in R$, therefore $(g, f), (f, h) \in R^s$ and thus $(g, h) \in (R^s)^+$. On the other hand $R^+ = R$ and therefore neither (g, h) nor (h, g) are in R^+ and so $(g, h) \notin (R^+)^s$.

The following holds $(R^s)^+ \supseteq (R^+)^s$. Consider a pair $(f, g) \in (R^+)^s$. From this we know that either (f, g) or (g, f) is in R^+. By the definition of transitive closure we know that one of the pairs is in R or there is a path whose consecutive elements are in R. Since R^s has more elements than R, we can deduce that either both pairs are in R^s or there is a path and its reverse which contains both pairs; therefore, both are contained in $(R^s)^+$. □

2. A vending machine dispensing books of stamps accepts only £1 coins, £1 notes and £5 notes.

(a) Find a recurrence relation for the number of ways to deposit £n in the vending machine, where the order in which the coins and notes are deposited matters.

(b) What are the initial conditions?

(c) How many ways are there to deposit £10 for a book of stamps?

Solution:

(a) Let a_n be the number of ways to deposit £n in the vending machine. We must express a_n in terms of earlier terms in the sequence. If we want to deposit £n, we may start with a pound coin and then deposit $n - 1$ pounds. This gives us a_{n-1} ways to deposit n pounds. We can also start with a pound note and then deposit $n - 1$ pounds. This gives us a_{n-1} more ways to deposit n pounds. Finally, we can deposit a five pound note and follow that with $n - 5$ pounds; there are a_{n-5} ways to do this. Therefore the recurrence relation is $a_n = 2a_{n-1} + a_{n-5}$. Note that this is valid for $n \geq 5$, since otherwise a_{n-5} makes no sense.

(b) We need initial conditions for all subscripts from 0 to 4. It is clear that $a_0 = 1$ (deposit nothing) and $a_1 = 2$ (deposit either the pound coin or the pound note). It is also not hard to see that $a_2 = 2^2 = 4$, $a_3 = 2^3 = 8$ and $a_4 = 2^4 = 16$, since each sequence of n coins and notes corresponds to a way to deposit n pounds.

(c) We will compute a_5 through a_{10} using the recurrence relation:

$$a_5 = 2a_4 + a_0 = 2 \cdot 16 + 1 = 33$$
$$a_6 = 2a_5 + a_1 = 2 \cdot 33 + 2 = 68$$
$$a_7 = 2a_6 + a_2 = 2 \cdot 68 + 4 = 140$$
$$a_8 = 2a_7 + a_3 = 2 \cdot 140 + 8 = 288$$
$$a_9 = 2a_8 + a_4 = 2 \cdot 288 + 16 = 592$$
$$a_{10} = 2a_9 + a_5 = 2 \cdot 592 + 33 = 1217$$

□

3. For this question you are not allowed to invoke any set that is known to be uncountable (such as subsets of $\mathbb{R}$) in your answer. Let $A = \{a, b, c\}$. Consider the set $F = \{f \mid f : \mathbb{Z}^+ \to A\}$: that is, F is the set of all functions from $\mathbb{Z}^+$ to A. Using diagonalization, prove that F is uncountable.

   Solution:
   Towards a contradiction, assume that the set F is countable. Therefore, there is a bijection $g : \mathbb{Z}^+ \to F$. So, g is surjective. Assume that $g(i) = f_i$ for $i \in Z^+$. So $F = \{f_1, \ldots, f_m, \ldots\}$. Let f be the function defined as follows: for any $n \in \mathbb{Z}^+$,

$$f(n) = \text{ if } f_n(n) = a \text{ then } b \text{ else } a$$

   Clearly, $f \in F$; however, $f \neq f_i$ for all i because $f(i) \neq f_i(i)$. Therefore, g is not surjective, which is a contradiction. $\square$

4. Determine (and prove) whether each of these sets is countably infinite or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence (i.e., bijection) between the set of positive integers and that set.

   (a) the odd negative integers

   (b) the real numbers in the open interval $(0, 2)$

   (c) the irrational numbers in the open interval $(0, 2)$

   (d) the set $A \times \mathbb{Z}^+$ where $A = \{2, 3\}$

   Solution:
   Let X be the set in question.

   (a) We prove that X is countably infinite by showing that the function $f : \mathbb{Z}^+ \to X$ defined by $f(x) = -2x + 1$ is a bijection. First we have to prove that the codomain of f is actually X. Let $x \in \mathbb{Z}^+$. Then, $x \geq 1$, so $-2x \leq -2$. Therefore $f(x) = -2x + 1 \leq -1$, which means that $f(x)$ is negative and clearly odd. To show that it is injective, let $f(x) = f(y)$. Then, $-2x + 1 = -2y + 1$, from which we get $x = y$. To show that it is surjective we take an arbitrary odd negative integer y. Let $x = -(y - 1)/2$. Clearly $f(x) = y$ so we just have to show that $x \in \mathbb{Z}^+$. Given that y is negative and odd, $y - 1$ is negative and even. Then, $(y - 1)/2$ is a negative integer, and thus $x = -(y - 1)/2$ is a positive integer.

   (b) From Cantor's diagonalization argument we know that the set of real numbers between 0 and 1 is uncountable. We call this set A. Then, we can prove that $|X| = |A|$ by showing that the function $f : A \to X$ given by $f(a) = 2a$ is a bijection. It is clearly injective because if $2a = 2b$ then $a = b$, and it is surjective because if $x \in X$ then $x/2 \in A$ and $f(x/2) = x$. Then, if X were countable A would also be countable, which is a contradiction.

   (c) The set of real numbers between 0 and 2 is the union of X and the set of rationals between 0 and 2 (we call this set A). The set of rationals between 0 and 2 is countable because there is an injection into $\mathbb{Q}$ (by inclusion). If X were countable then $X \cup A$ would also be countable, which contradicts the result of (b).

   (d) Let $f : A \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

$$f(a, x) = \begin{cases} 2x & \text{if } a = 2 \\ 2x - 1 & \text{if } a = 3 \end{cases}$$

   To prove that this function is injective, let $f(a, x) = f(b, y)$. If this number is even, then $a = b = 2$, which means that $f(a, x) = f(2, x) = 2x$ and $f(b, y) = f(2, y) = 2y$. Then,

$2x = 2y$, which implies that $x = y$ and therefore $(a, x) = (b, y)$. If the number is odd then $a = b = 3$, which means that $f(a, x) = f(3, x) = 2x - 1$ and $f(b, y) = f(3, y) = 2y - 1$. Then, $2x - 1 = 2y - 1$, which implies that $x = y$ and therefore $(a, x) = (b, y)$. To prove that f is surjective, let $y \in \mathbb{Z}^+$. If y is even then let $(a, x) = (2, y/2)$, which is clearly an element of $A \times \mathbb{Z}^+$. Then, $f(a, x) = 2(y/2) = y$. If y is odd then let $(a, x) = (3, (y+1)/2)$, which is clearly an element of $A \times \mathbb{Z}^+$. Then, $f(a, x) = 2(y + 1)/2 - 1 = y$.

$\square$

5. Prove that for all sets A if $A \subseteq \mathbb{Z}^+$ then either A is finite or $|A| = |\mathbb{Z}^+|$.

Solution:

Assume that $A \subseteq \mathbb{Z}^+$, that is A only contains positive integers. If A is finite, that is for some $n \geq 0$, $|A| = n$ then we are done. Otherwise A is an infinite set of positive integers. We now need to show that there is a bijection $f : \mathbb{Z}^+ \to A$. So, we need to associate an element of A with each positive integer. One way to do this is to consider the relative ordering of elements in A. For each $a \in A$, let the index of a in A be $i(a) = |\{b \in A \mid b \leq a\}|$. For each $a \in A$, the index of a is finite (because A consists of positive integers, so $i(a) \leq a$) and is unique (because $i(a) = i(b)$ iff $a = b$). As A is infinite for each $z \in \mathbb{Z}^+$ there is an $a \in A$ such that $z = i(a)$. Therefore, the function $f : \mathbb{Z}^+ \to A$ which maps z to $a \in A$ such that $i(a) = z$ is a bijection: clearly it is both injective and surjective, as required. $\square$