DMMR Tutorial sheet 2

Sets, Functions, Relations (part 1)

September 26th, 2019

1. (a) Prove the set absorption law $A \cup (A \cap B) = A$.

   Solution:
   We show that $A \cup (A \cap B) \subseteq A$ and $A \cup (A \cap B) \supseteq A$.

   - For the first consider an element x in $A \cup (A \cap B)$. From the definition of $\cup$ we know that either $x \in A$ or $x \in (A \cap B)$. In the first case we are done. In the other case we know that x is both in A and in B from the definition of $\cap$. Therefore we get $x \in A$ in all cases.
   - Consider an element $x \in A$. From the definition of $\cup$ we immediately get $x \in A \cup (A \cap B)$.

   □

   (b) Prove the set distribution law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   Solution:
   Similarly we show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$.

   - For the first, consider an element $x \in A \cup (B \cap C)$. So, $x \in A$ or $x \in (B \cap C)$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$; so, $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$ then $x \in B$ and $x \in C$; so, $x \in A \cup B$ and $x \in A \cup C$ and, therefore, $x \in (A \cup B) \cap (A \cup C)$.
   - If $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup B$ and $x \in A \cup C$; so, $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$; so $x \in A$ or ($x \in B$ and $x \in C$); so, $x \in A \cup (B \cap C)$.

   □

   (c) Prove the following set identity $(B - A) \cup (C - A) = (B \cup C) - A$

   Solution:
   Again we show $(B - A) \cup (C - A) \subseteq (B \cup C) - A$ and $(B - A) \cup (C - A) \supseteq (B \cup C) - A$.

   - For the first, consider an element $x \in (B - A) \cup (C - A)$. So, $x \in B - A$ or $x \in C - A$; so, ($x \in B$ and $x \notin A$) or ($x \in C$ and $x \notin A$); consequently, $x \in B$ or $x \in C$ and $x \notin A$; so, $x \in (B \cup C) - A$.
   - If $x \in (B \cup C) - A$ then $x \in B \cup C$ and $x \notin A$; so, $x \in B$ or $x \in C$ and $x \notin A$; therefore, $x \in B - A$ or $x \in C - A$; so, $x \in (B - A) \cup (C - A)$.

   □

2. Let A, B, C be sets. Derive a formula for $|A \cup B \cup C|$, which only uses the cardinality $|\cdot|$, intersection $\cap$ and arithmetic operators.

   Solution:

$$
\begin{aligned}
|A \cup B \cup C| &= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + (|B| + |C| - |B \cap C|) - |(A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap B)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(B \cap A) \cap (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |B \cap (A \cap (A \cap C))| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |B \cap ((A \cap A) \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |B \cap A \cap C|
\end{aligned}
$$

□

3. (a) Determine whether the function $f : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}$ is surjective if

   i. $f(m, n) = m^2 + n^2$
   ii. $f(m, n) = m$
   iii. $f(m, n) = |n|$
   iv. $f(m, n) = m - n$

   Solution:

   i. The function is not surjective because not every integer is the sum of two perfect squares. For example $-|n|$ and 3 are not the sum of two perfect squares (for any n).
   ii. The function is surjective because for any $z \in \mathbb{Z}$ we can choose a pair $(z, x) \in \mathbb{Z} \times \mathbb{Z}$ and $f(z, x) = z$.
   iii. The function is not surjective because $|n|$ is always positive, so there exists no (x, y) such that $f(x, y) = -|n|$.
   iv. The function is surjective because for every z integer $f(z, 0) = z - 0 = z$.

   □

   (b) Assume functions $g : A \to B$ and $f : B \to C$. Prove or disprove the following statements.

   i. If $f \circ g$ and g are injective then f is injective.
   Solution:
   This statement is not correct; let $A = \{a, b\} = C$ and $B = \{a, b, c\}$; let $g(a) = a$ and $g(b) = b$; and $f(a) = a$; $f(b) = b$ and $f(c) = a$. Now $f \circ g$ is injective since $(f \circ g)(a) \neq (f \circ g)(b)$; similarly g is injective; however, f is not injective because $f(a) = f(c)$. □
   ii. If $f \circ g$ and f are injective then g is injective.
   Solution:
   This statement is true. In fact, we prove the slightly stronger: if $f \circ g$ is injective then g is injective. By way of contradiction assume $f \circ g$ is injective and g is not. So, for some $a, a' \in A$, $a \neq a'$ and $g(a) = g(a')$; so, $f(g(a)) = f(g(a'))$, so $(f \circ g)(a) = (f \circ g)(a')$ which contradicts that $f \circ g$ is injective. □

4. Given function $f : A \to B$, we define the function $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$ as follows: $P_f(A') = \{b \in B \mid \exists a \in A'(f(a) = b)\}$ for $A' \subseteq A$. Prove the following statements.

(a) $f : A \to B$ is injective iff $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$ is injective.

Solution:

First assume $f : A \to B$ is injective; so for any $a, a' \in A$ if $a \neq a'$ then $f(a) \neq f(a')$. Consider $A' \subseteq A$ and $A'' \subseteq A$ and assume $A' \neq A''$: we need to show that $P_f(A') \neq P_f(A'')$. Without loss of generality assume $a \in A'$ and $a \notin A''$. By definition $f(a) \in P_f(A')$ but $f(a) \notin P_f(A'')$ as otherwise $f(a) = f(a')$ for some $a' \in A''$ with $a \neq a'$.

For the other direction assume $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$ is injective; so for any $A', A'' \subseteq A$ if $A' \neq A''$ then $P_f(A') \neq P_f(A'')$. To show $f : A \to B$ is injective, consider $a, a' \in A$ where $a \neq a'$; we know $P_f(\{a\}) \neq P_f(\{a'\})$; so $f(a) \neq f(a')$. $\square$

(b) $f : A \to B$ is surjective iff $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$ is surjective.

Solution:

First assume $f : A \to B$ is surjective; so for every element $b \in B$ there is an $a \in A$ with $b = f(a)$. Consider $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$; it is surjective if for every $B' \subseteq B$ there is $A' \subseteq A$ such that $P_f(A') = B'$. Let $B' \subseteq B$ and let $A' = \{a \in A \mid \exists b \in B' (f(a) = b)\}$; since f is surjective, for every element $b \in B'$ there is an $a \in A'$ such that $f(a) = b$, so $P_f(A') = B'$.

For the other direction assume $P_f : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})$ is surjective; so for every $B' \subseteq B$ there is an A' such that $P_f(A') = B'$. Consider the full set B; there is an $A' \subseteq A$ such that $P_f(A') = B$; so for every $b \in B$ there is an $a \in A' \subseteq A$ such that $f(a) = b$; so f is surjective. $\square$

5. For each of the following relations on the set of all real numbers, determine whether it is reflexive, symmetric, antisymmetric, and/or transitive, where (x, y) are related if and only if

(a) $x - y$ is a rational number.

(b) $x = 2y$.

(c) $xy \geq 0$.

(d) $xy = 0$.

(e) $x = 1$.

(f) $x = 1$ or $y = 1$.

Solution:

(a) **Reflexive**: Yes, because $x - x = 0 \in \mathbb{Q}$ **Symmetric**: Yes, because if $x - y$ is rational then $-(x - y) = y - x$ is also rational. **Antisymmetric**: No, because $2 - 1$ is rational and $1 - 2$ is rational, but $1 \neq 2$. **Transitive**: Yes, because if $x - y$ and $y - z$ are both rational, then $x - z = (x - y) + (y - z)$ is also rational.

(b) **Reflexive**: No, because $1 \neq 2 \cdot 1$. **Symmetric**: No. Let $x = 2$ and $y = 1$. Then, $x = 2 \cdot y$, but $y \neq 2 \cdot x$. **Antisymmetric**: Yes, because if $x = 2y$ and $y = 2x$ then $x = 4x$, which means that $x = 0 = y$. **Transitive**: No. Let $x = 4$, $y = 2$ and $z = 1$. Then, $x = 2 \cdot y$ and $y = 2 \cdot z$ but $x \neq 2 \cdot z$.

(c) **Reflexive**: Yes, because $x^2 \geq 0$. **Symmetric**: Yes, because if $xy \geq 0$ then $yx = xy \geq 0$. **Antisymmetric**: No, because $1 \cdot 2 = 2 \cdot 1 \geq 0$, but $2 \neq 1$. **Transitive**: No. Let $x = -1$, $y = 0$ and $z = 1$. Then, $xy = (-1) \cdot 0 \geq 0$ and $yz = 0 \cdot 1 \geq 0$ but $xz = (-1) \cdot 1 = -1 < 0$.

(d) **Reflexive**: No, because $1 \cdot 1 \neq 0$. **Symmetric**: Yes, because if $xy = 0$ then $yx = xy = 0$. **Antisymmetric**: No. Let $x = 1$ and $y = 0$. Then $xy = 0 = yx$, but $0 \neq 1$. **Transitive**: No. Let $x = z = 1$ and $y = 0$. Then, $xy = 0 = yz$ but $xz = 1 \neq 0$.

(e) **Reflexive**: No. Let $x = 2$. Then, (x, x) is not in the relation. **Symmetric**: No. $(1, 2)$ is in the relation, but $(2, 1)$ is not. **Antisymmetric**: Yes, because if (x, y) and (y, x) both are in the relation then $x = 1 = y$. **Transitive**: Yes, because if (x, y) and (y, z) are both in the relation, then $x = 1$, which means that $(x, z) = (1, z)$, which is in the relation.

(f) **Reflexive**: No, because $(2, 2)$ is not in the relation. **Symmetric**: Yes, because if $x = 1 \vee y = 1$ then $y = 1 \vee x = 1$. **Antisymmetric**: No, because $(1, 2)$ and $(2, 1)$ are in the relation, but $2 \neq 1$. **Transitive**: No. Let $x = z = 2$ and $y = 1$. Then, $(x, y) = (2, 1)$ and $(1, 2)$ are in the relation, but $(x, z) = (2, 2)$ is not.

$\square$