DMMR Tutorial sheet 8

More on Graphs, Trees

November 8, 2019

1. Describe how to extend Dijkstra's algorithm so that, given a directed edge-weighted graph $G = (V, E, w)$, and given a source vertex $s \in V$ and another vertex $t \in V$, $s \neq t$, the algorithm computes not just the *length* of a shortest path from s to t (assuming such a directed path exists), but also outputs an actual shortest path (as a sequence of edges) from s to t.

2. For a (simple, undirected) graph $G = (V, E)$, let $d(G)$ denote the maximum degree of any vertex of G. The chromatic number, $\chi(G)$, of a (simple, undirected) graph $G = (V, E)$, is the smallest number of colours with which we can colour all vertices in V such that no two adjacent vertices have the same colour.

   - Prove that every graph, $G = (V, E)$ has chromatic number $\chi(G) \leq d(G) + 1$.
   - Argue that your proof actually provides an efficient algorithm to obtain a colouring of G with at most $d(G) + 1$ colours.
   - Show that both the complete graph, K_n, and the k-cycle, C_k, with $n, k \in \mathbb{N}^{>1}$ and k odd, are examples of graphs where $d(G) + 1$ colours are necessary i.e., where $\chi(G) = d(G) + 1$.

   (Food for thought: are there any other connected graphs where $d(G) + 1$ colours are necessary?)

3. Let $m > 1$ and h be positive integers. A complete m-ary tree is a rooted full m-ary tree in which every leaf has the same distance from the root. How many vertices, how many leaves, and how many internal vertices, does a complete m-ary tree of height h have? Prove why these formulas are correct.

4. A single weighing on a balance scale can be used to compare the weight of any two sets of coins, A and B, and to determine whether A weights: (i) more than B, (ii) less than B, or (iii) the same as B. Suppose you are given 4 gold coins, and another 3 silver coins, and you are told that:

   - exactly one of the gold coins is counterfeit.
   - all authentic gold coins weigh the same, and the three silver coins also each weigh the same as an authentic gold coin.
   - the counterfeit gold coin is either heavier or lighter than the others.

   Use decision tree arguments to determine the minimum number, k, of weighings that is necessary and sufficient, in the worst case, in order to determine BOTH which gold coin is counterfeit AND whether that coin is heavier or lighter than the others. Justify why your answer is correctness. Give an algorithm (a decision tree) which uses at most k weighings to find the correct answer.

5. Prove that any (simple, undirected) graph $G = (V, E)$, with $m = |E|$ edges has chromatic number $\chi(G) \leq \sqrt{2m} + 1$.