1. Suppose there is a finite set $A$ of job applicants and a finite set $J$ of job openings, and that for some fixed positive integer $k \geq 1$, every job applicant $a \in A$ has applied to exactly $k$ jobs in $J$, and every job opening $j \in J$ has received exactly $k$ job applications from applicants in $A$.

Prove that $|A| = |J|$, and that it is possible to match each job applicant $a \in A$ with a unique job $f(a) \in J$ which $a$ has applied for, such that all applicants and all jobs are “matched”, and no job (no applicant) is matched to more than one applicant (one job, respectively). In other words, prove that there is a bijective function $f : A \to J$, such that, for all $a \in A, a$ has applied to $f(a)$.

[Hint: apply the generalized pigeonhole principle to show that Hall’s Theorem applies in this setting.]

2. How many non-isomorphic (simple, undirected) graphs are there with exactly 4 vertices? Justify your answer.

3. Suppose $G = (V, E)$ is a directed graph, and $u$ and $v$ are vertices of $G$. Show that either $u$ and $v$ are in the same strongly connected component of $G$, or they are in disjoint strongly connected components of $G$.

4. Recall that the $n$-dimensional hypercube, or $n$-cube, is the simple undirected graph whose nodes are bit strings of length $n$, and such that there is an edge between a pair of nodes if and only if their bit strings differ in exactly one bit position.

   (a) For what values of $n \geq 1$ does the $n$-cube have an Euler circuit?

   (b) Prove by induction that for all $n \geq 2$, the $n$-cube has a Hamiltonian circuit.

5. Recall that a (simple, undirected) graph $G = (V, E)$ is called bipartite if the vertices $V$ can be partitioned into two sets, $V_1$ and $V_2$ (such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$), such that for all edges $e = \{u, v\} \in E$, $V_1 \cap e \neq \emptyset$ and $V_2 \cap e \neq \emptyset$. (In other words, all edges are between a vertex in $V_1$ and a vertex in $V_2$.)

Recall that a cycle (or circuit) in a graph $G = (V, E)$ is a path of length $k \geq 1$ that starts and ends in the same vertex. The length of a path is the number of edges on it.

Prove that a graph $G$ is bipartite if and only if $G$ does not contain any cycles of odd length.

[Hint: if no odd cycle exists in a connected graph, use that to bipartition its vertices according to whether the length of the shortest path from a given vertex is odd or even.]

Your solution (to the last question on the sheet) must be handed in on paper at the ITO by Wednesday, 9 November, at 4:00pm.