1. Analogous to the definition of \( \gcd \) we define the least common multiple (lcm) in the following way:
   For two numbers \( a \) and \( b \) with the prime factorisation \( a = p_1^{a_1} \cdots p_n^{a_n} \), \( b = p_1^{b_1} \cdots p_n^{b_n} \) we define
   \[
   \text{lcm}(a, b) := p_1^{\max(a_1, b_1)} \cdots p_n^{\max(a_n, b_n)}
   \]
   Show that if \( a \) and \( b \) are positive integers, then \( ab = \gcd(a, b) \cdot \text{lcm}(a, b) \).

2. Use the Euclidean Algorithm to find
   (a) \( \gcd(12, 18) \)
   (b) \( \gcd(111, 201) \)
   (c) \( \gcd(1001, 1331) \)
   (d) \( \gcd(12345, 54321) \)
   (e) \( \gcd(1000, 5040) \)
   (f) \( \gcd(9888, 6060) \)

3. This question uses Fermat’s little theorem.
   (a) Use Fermat’s little theorem to compute \( 3^{304} \mod 11 \) and \( 3^{304} \mod 13 \)
   (b) Show with the help of Fermat’s little theorem that if \( n \) is a positive integer, then \( 42 \) divides \( n^7 - n \).

4. (a) Find the least integer \( n \) such that \( f(x) \) is \( O(x^n) \) for each of these functions.
   i. \( f(x) = 2x^3 + x^4 \log x \)
   ii. \( f(x) = 3x^3 + (\log x)^4 \)
   iii. \( f(x) = (x^4 + x^2 + 1)/(x^3 + 1) \)
   (b) Let \( f_1(x) \) and \( f_2(x) \) be functions from the set of real numbers to the set of positive real numbers. Show that if \( f_1(x) \) and \( f_2(x) \) are both \( \Theta(g(x)) \), where \( g(x) \) is a function from the set of real numbers to the set of positive real numbers, then \( f_1(x) + f_2(x) \) is \( \Theta(g(x)) \). Is this still true if \( f_1(x) \) and \( f_2(x) \) can take negative values?
5. Using the Chinese Remainder Theorem, find a solution to the following system of equivalences.

\[
x \equiv 1 \pmod{2} \\
x \equiv 2 \pmod{3} \\
x \equiv 3 \pmod{5} \\
x \equiv 4 \pmod{11}
\]

Explain your calculations.

Solutions (to the last question on the sheet) must be handed in on paper at the ITO by Wednesday, 26 October, 4:00pm. Please post it into the grey metal box on the wall outside the ITO.