DMMR Tutorial sheet 3

Relations (part 2), Recurrences, Cardinality

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1. Many program analysis methods rely on call graphs. A call graph is a binary relation R on function (or method) names in a program. A pair $(f, g) \in R$ when the body of function (method) f in the program calls the function (method) g. For example, consider the following abstracted code for a function (method) f where we have taken out all the parameters.

```
function f() {
  g();
  h()
}
```

   This means that in the body of f both g and h are called. So, the pairs $(f, g) \in R$ and $(f, h) \in R$. In turn the functions (methods) g, h may call other functions (methods).

   The transitive closure of relation R, written R^+, is the following binary relation: $(f, g) \in R^+$ iff $(f, g) \in R$ or there is $n \geq 1$ such that $(f, f_1) \in R, (f_1, f_2) \in R, \ldots, (f_n, g) \in R$. That is, there is a path from f to g of consecutive pairs from R.

   The symmetric closure of relation R, written R^s, is the following binary relation: $(f, g) \in R^s$ iff $(f, g) \in R$ or $(g, f) \in R$.

   (a) Prove that R^+ is transitive.

   (b) Explain what information the relations $(R^s)^+$ and $(R^+)^s$ contain about the function (method) calls in the program.

   (c) Decide which of the two relations $(R^s)^+$ and $(R^+)^s$ subsumes the other, give a formal proof of your claim and show an example relation R and a pair of functions which is contained in only one of them.

2. A vending machine dispensing books of stamps accepts only £1 coins, £1 notes and £5 notes.

   (a) Find a recurrence relation for the number of ways to deposit £n in the vending machine, where the order in which the coins and notes are deposited matters.

   (b) What are the initial conditions?

   (c) How many ways are there to deposit £10 for a book of stamps?

3. For this question you are not allowed to invoke any set that is known to be uncountable (such as subsets of $\mathbb{R}$) in your answer. Let $A = \{a, b, c\}$. Consider the set $F = \{f \mid f : \mathbb{Z}^+ \to A\}$: that is, F is the set of all functions from $\mathbb{Z}^+$ to A. Using diagonalization, prove that F is uncountable.

4. Determine (and prove) whether each of these sets is countably infinite or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence (i.e., bijection) between the set of positive integers and that set.

(a) the odd negative integers
(b) the real numbers in the open interval $(0, 2)$
(c) the irrational numbers in the open interval $(0, 2)$
(d) the set $A \times \mathbb{Z}^+$ where $A = \{2, 3\}$

5. Prove that for all sets A if $A \subseteq \mathbb{Z}^+$ then either A is finite or $|A| = |\mathbb{Z}^+|$.