DMMR Tutorial sheet 2

Sets, Functions, Relations (part 1)

September 26th, 2019

1. (a) Prove the set absorption law $A \cup (A \cap B) = A$.
   (b) Prove the set distribution law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (c) Prove the following set identity $(B - A) \cup (C - A) = (B \cup C) - A$

2. Let A, B, C be sets. Derive a formula for $|A \cup B \cup C|$, which only uses the cardinality $|\cdot|$, intersection $\cap$ and arithmetic operators.

3. (a) Determine whether the function $f : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}$ is surjective if

   i. $f(m, n) = m^2 + n^2$
   ii. $f(m, n) = m$
   iii. $f(m, n) = |n|$
   iv. $f(m, n) = m - n$

   (b) Assume functions $g : A \to B$ and $f : B \to C$. Prove or disprove the following statements.
   i. If $f \circ g$ and g are injective then f is injective.
   ii. If $f \circ g$ and f are injective then g is injective.

4. Given function $f : A \to B$, we define the function $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$ as follows: $P_f(A') = \{b \in B \mid \exists a \in A'(f(a) = b)\}$ for $A' \subseteq A$. Prove the following statements.

   (a) $f : A \to B$ is injective iff $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$ is injective.
   (b) $f : A \to B$ is surjective iff $P_f : \mathcal{P}(A) \to \mathcal{P}(B)$ is surjective.

5. For each of the following relations on the set of all real numbers, determine whether it is reflexive, symmetric, antisymmetric, and/or transitive, where (x, y) are related if and only if

   (a) $x - y$ is a rational number.
   (b) $x = 2y$.
   (c) $xy \geq 0$.
   (d) $xy = 0$.
   (e) $x = 1$.
   (f) $x = 1$ or $y = 1$.