# Asymmetric ciphers

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# Introduction

So far: how two users can protect data using a shared secret key

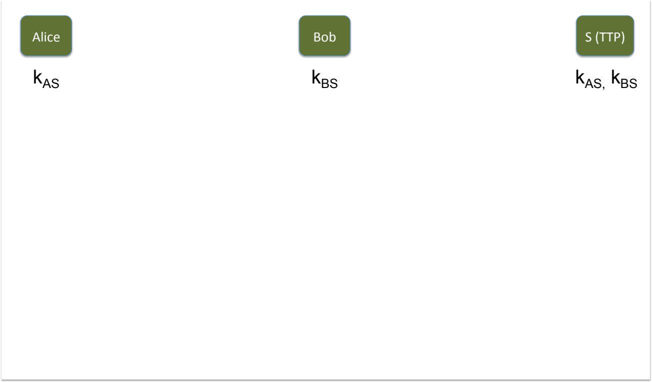
- One shared secret key per pair of users that want to communicate

Our goal now: how to establish a shared secret key to begin with?

- Trusted Third Party (TTP)
- Diffie-Hellman (DH) protocol
- *RSA*
- ElGamal (EG)

# Online Trusted Third Party (TTP)

- Users $U_1, U_2, U_3, \ldots, U_n, \ldots$
- Each user $U_i$ has a shared secret key $K_i$ with the TTP
- $U_i$ and $U_j$ can establish a key $K_{i,j}$ with the help of the TTP
  ex: using Paulson's variant of the Yahalom protocol



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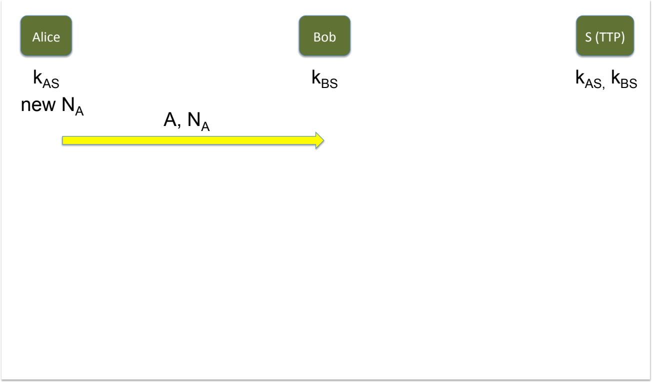

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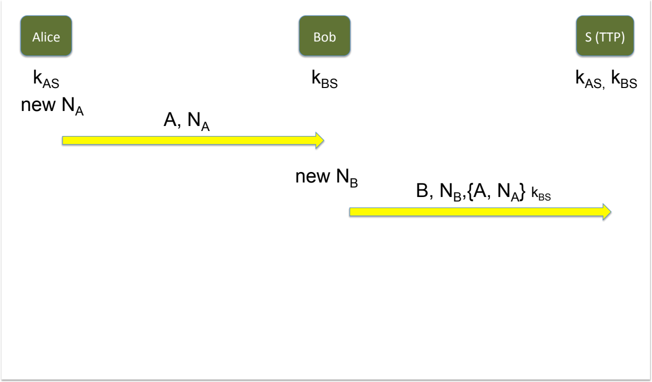

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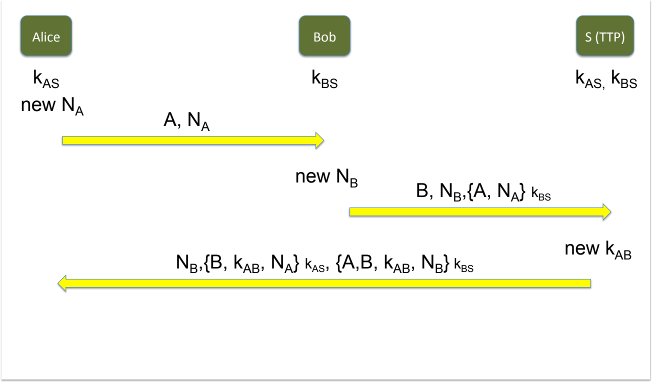

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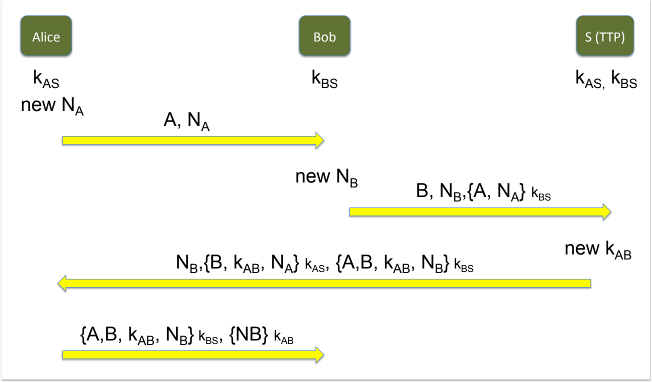

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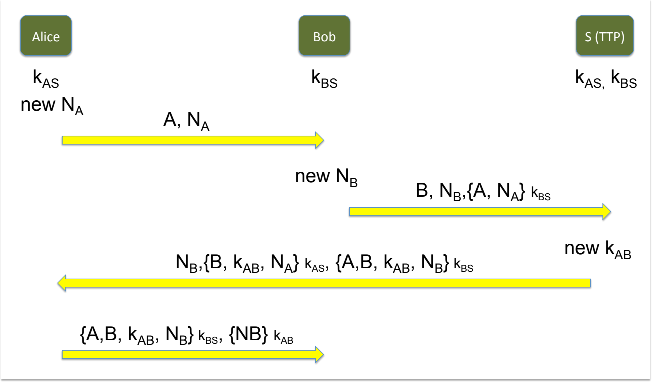
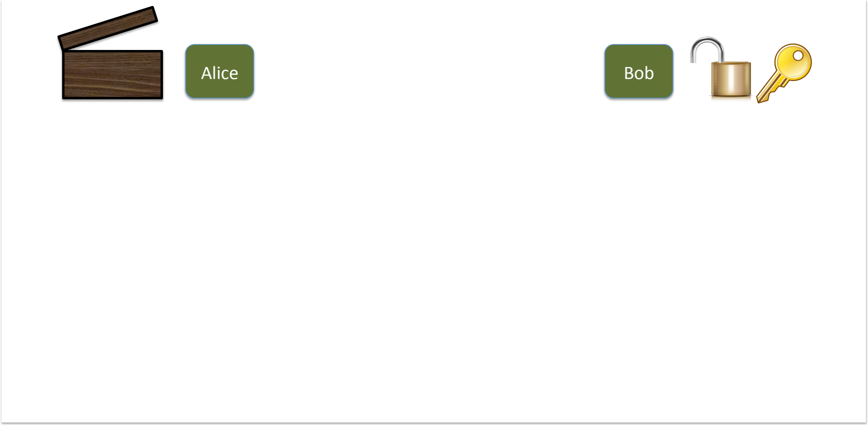



Question: can we establish a shared secret key without a TTP?
Answer: Yes!

# Public-key encryption in pictures



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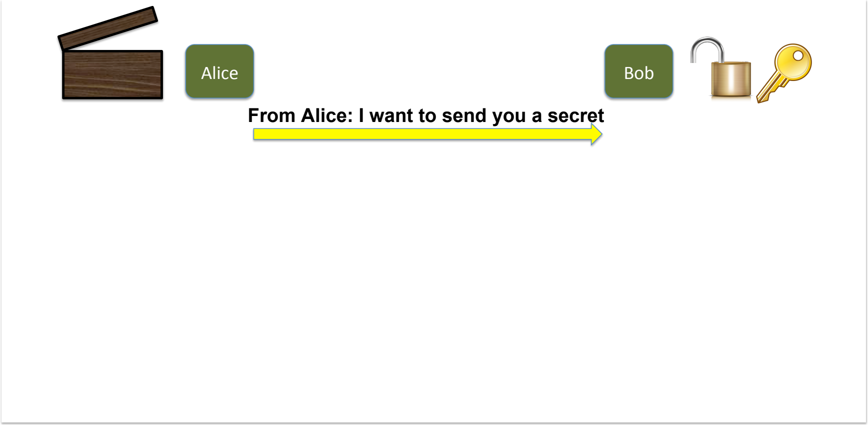

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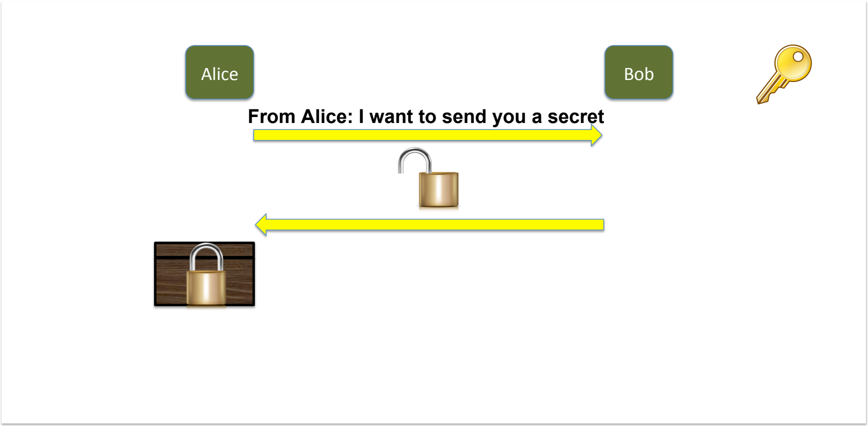

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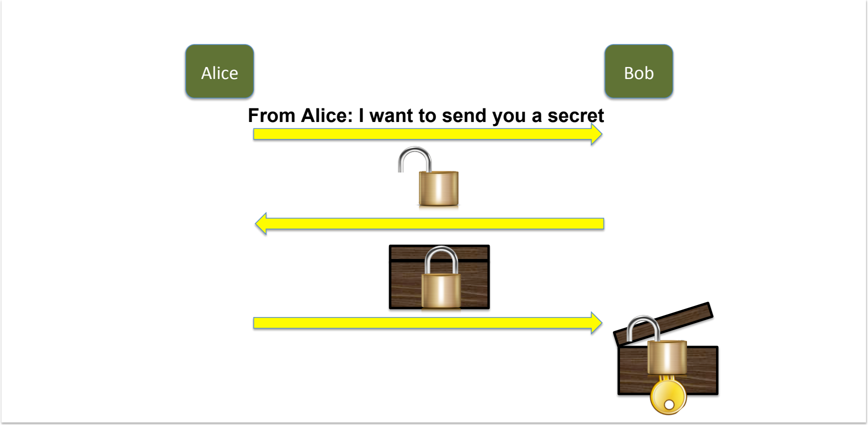
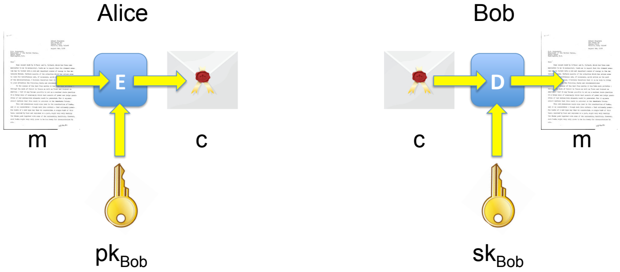


# Public-key encryption

- key generation algorithm: $G : \to \mathcal{K} \times \mathcal{K}$
  encryption algorithm $E : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$
  decryption algorithm $D : \mathcal{K} \times \mathcal{C} \to \mathcal{M}$
  st. $\forall (sk, pk) \in G$, and $\forall m \in \mathcal{M}$, $D(sk, E(pk, m)) = m$




- the decryption key $sk_{Bob}$ is secret (only known to Bob). The encryption key $pk_{Bob}$ is known to everyone. And $sk_{Bob} \neq pk_{Bob}$

# We need a bit of number theory now

# Primes

**Definition**

$p \in \mathbb{N}$ is a **prime** if its only divisors are 1 and $p$

Ex: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

**Theorem**

*Every $n \in \mathbb{N}$ has a* **unique factorization** *as a product of prime numbers (which are called its factors)*

Ex: $23244 = 2 \times 2 \times 3 \times 13 \times 149$

# Relative primes

**Definition**

$a$ and $b$ in $\mathbb{Z}$ are **relative primes** if they have no common factors

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The Euler function $\phi(n)$ is the number of elements that are relative primes with $n$:

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- For $p$ and $q$ primes: $\phi(p \cdot q) =$

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- For $p$ prime: $\phi(p) = p\text{-}1$
- For $p$ and $q$ primes: $\phi(p \cdot q) = (p\text{-}1)(q\text{-}1)$

# $\mathbb{Z}_n$

- Let $n \in \mathbb{N}$. We define $\mathbb{Z}_n = \{0, \dots n\text{-}1\}$

$$\forall a \in \mathbb{Z},\ \forall b \in \mathbb{Z}_n,\ a \equiv b \pmod{n} \ \Leftrightarrow\ \exists k \in \mathbb{N}.\ a = b + k \cdot n$$

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**Theorem**

*Let $n \in \mathbb{N}$. Let $x \in \mathbb{Z}_n$. $x$ has a inverse in $\mathbb{Z}_n$ iff* $\gcd(x, n) = 1$

# $(\mathbb{Z}_N)^*$

- Let $n \in \mathbb{N}$. We define $(\mathbb{Z}_n)^* = \{x \in \mathbb{Z}_n \mid \gcd(x, n) = 1\}$
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**Theorem (Euler)**

$\forall n \in \mathbb{N}$, $\forall x \in (\mathbb{Z}_n)^*$, $x^{\phi(n)} \equiv 1 \pmod{n}$

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**Theorem (Euler)**

*$\forall p$ prime, $(\mathbb{Z}_p)^*$ is a cyclic group, i.e.*

$$\exists g \in (\mathbb{Z}_p)^*, \ \{g, g^2, g^3, \ldots, g^{p-2}\} = (\mathbb{Z}_p)^*$$

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- DHP:
  input: prime $p$, generator $g$ of $(\mathbb{Z}_p)^*$, $g^a \pmod{p}$, $g^b \pmod{p}$
  output: $g^{ab} \pmod{p}$

**We can now go back and see how to establish a key without a TTP**

# The Diffie-Hellman (DH) protocol

- Assumption: the DHP is hard in $(\mathbb{Z}_p)^*$
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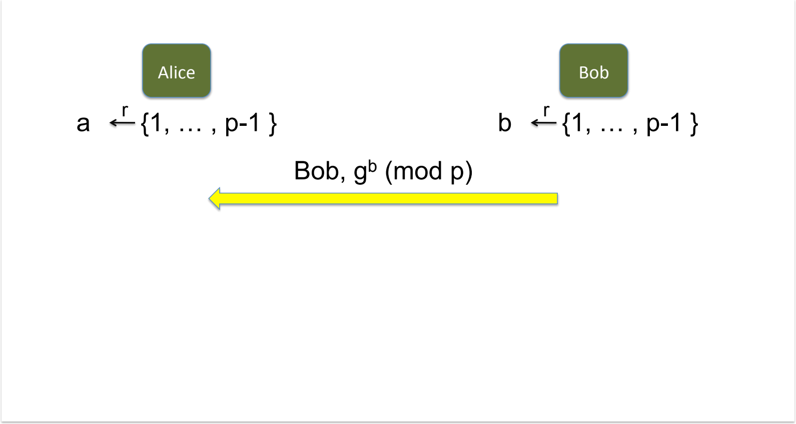

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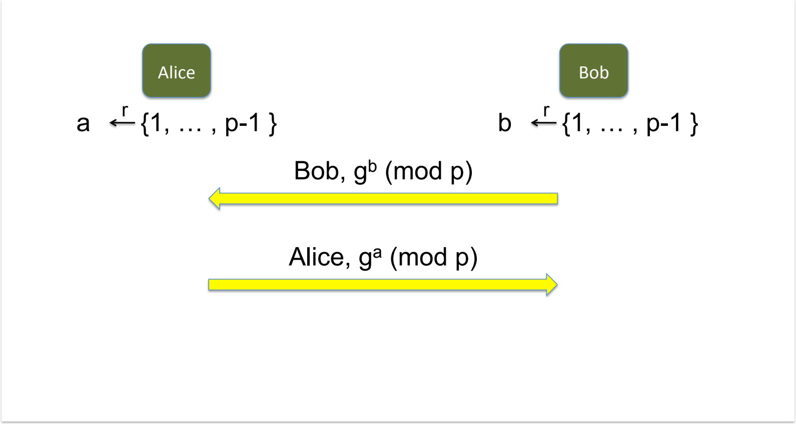

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Alice | Bob
--- | ---
$a \xleftarrow{r} \{1, \ldots, p\text{-}1\}$ | $b \xleftarrow{r} \{1, \ldots, p\text{-}1\}$

Bob, $g^b \pmod{p}$ ←

Alice, $g^a \pmod{p}$ →

$$k_{AB} = (g^b)^a \,[\mathrm{mod}\ p] = g^{ab} \pmod{p}$$

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# Man in the middle attack on DH

| Alice | Attacker | Bob |
|---|---|---|
| $a \xleftarrow{r} \{1, \ldots, p-1\}$ | $a' \xleftarrow{r} \{1, \ldots, p-1\}$ | $b \xleftarrow{r} \{1, \ldots, p-1\}$ |
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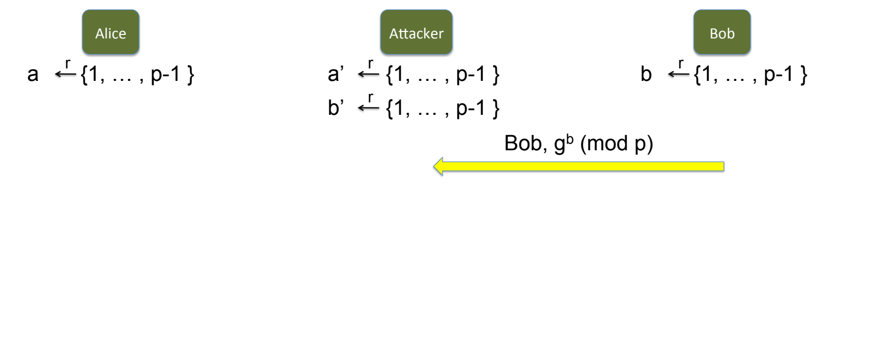

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Bob, $g^{b'}$ (mod p) $\longleftarrow$

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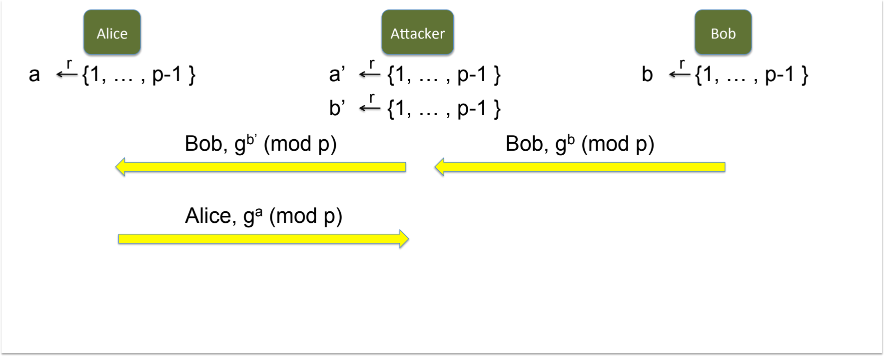

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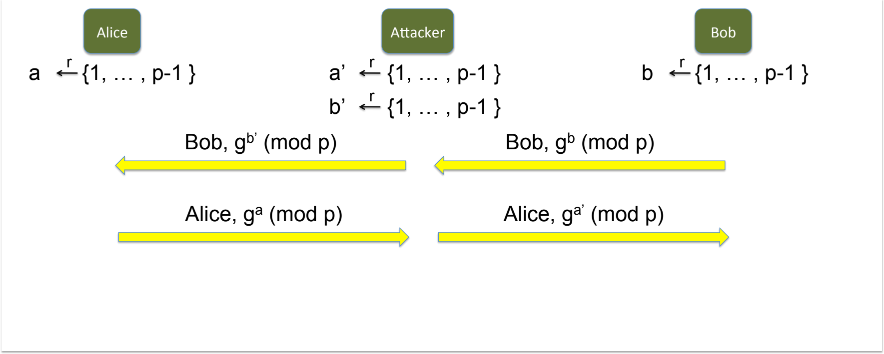

# Man in the middle attack on DH

**Alice** | **Attacker** | **Bob**

Alice: $a \xleftarrow{r} \{1, \ldots, p-1\}$

Attacker: $a' \xleftarrow{r} \{1, \ldots, p-1\}$, $b' \xleftarrow{r} \{1, \ldots, p-1\}$

Bob: $b \xleftarrow{r} \{1, \ldots, p-1\}$

Attacker → Alice: Bob, $g^{b'} \pmod p$

Bob → Attacker: Bob, $g^{b} \pmod p$

Alice → Attacker: Alice, $g^{a} \pmod p$

Attacker → Bob: Alice, $g^{a'} \pmod p$

Alice: $k_{AB} = g^{ab'} \pmod p$

Attacker:

$$k_B = (g^b)^{a'} = g^{ba'}$$
$$k_A = (g^{a'})^b = g^{a'b}$$

Bob: $k_{AB} = g^{a'b} \pmod p$

# *RSA* trapdoor permutation

- $G_{RSA}() = (pk, sk)$ where $pk = (N, e)$ and $sk = (N, d)$ and $N = p \cdot q$ with $p, q$ random primes and $e, d \in \mathbb{Z}$ st. $e \cdot d \equiv 1 \pmod{\phi(N)}$

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- Consistency: $\forall (pk, sk) = G_{RSA}(), \forall x,$ $RSA^{-1}(sk, RSA(pk, x)) = x$
  Proof: Let $pk = (N, e)$ and $sk = (N, d)$

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&= \quad x \cdot x^{k\phi(N)} \pmod{N} \\
&= \quad x \cdot (x^{\phi(N)})^k \pmod{N} \\
&\overset{\text{Euler}}{=} \quad x \pmod{N}
\end{aligned}
$$

# How NOT to use *RSA*

$(G_{RSA}, RSA, RSA^{-1})$ is called raw *RSA*. Do not use raw *RSA* directly as an asymmetric cipher!

*RSA* is deterministic $\Rightarrow$ not secure against chosen plaintext attacks

(Details on the board)

# ISO standard

Goal: build a CPA secure asymmetric cipher using $(G_{RSA}, RSA, RSA^{-1})$

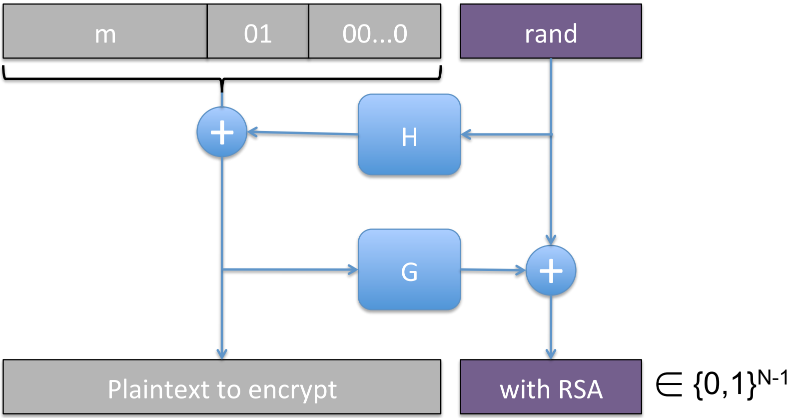
Let $(E_s, D_s)$ be a symmetric encryption scheme over $(\mathcal{M}, \mathcal{C}, \mathcal{K})$
Let $H: (\mathbb{Z}_N)^* \to \mathcal{K}$

Build $(G_{RSA}, E_{RSA}, D_{RSA})$ as follows

- $G_{RSA}()$ as described above
- $E_{RSA}(pk, m)$:
  - pick random $x \in (\mathbb{Z}_N)^*$
  - $y \leftarrow RSA(pk, x)(= x^e)$
  - $k \leftarrow H(x)$
  - $E_{RSA}(pk, m) = y || E_s(k, m)$
- $D_{RSA}(pk, y||c) = D_s(H(RSA^{-1}(sk, y)), c)$

# PKCS1 v2.0: *RSA*-OAEP

Goal: build a CPA secure asymmetric cipher using $(G_{RSA}, RSA, RSA^{-1})$



# ElGamal (EG)

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