Asymmetric ciphers

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Introduction

So far: how two users can protect data using a shared secret key

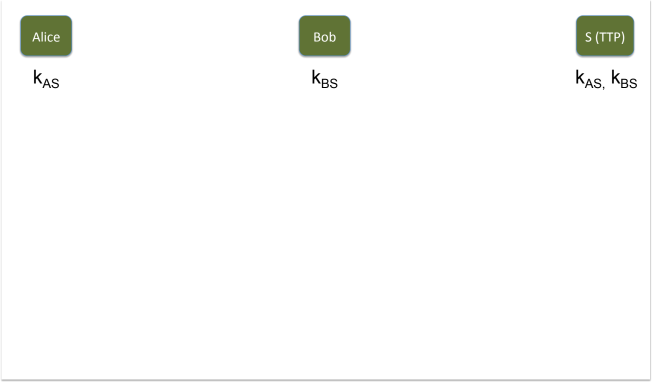
- One shared secret key per pair of users that want to communicate

Our goal now: how to establish a shared secret key to begin with?

- Trusted Third Party (TTP)
- Diffie-Hellman (DH) protocol
- *RSA*
- ElGamal (EG)

Online Trusted Third Party (TTP)

- Users $U_1, U_2, U_3, \ldots, U_n, \ldots$
- Each user U_i has a shared secret key K_i with the TTP
- U_i and U_j can establish a key $K_{i,j}$ with the help of the TTP
  ex: using Paulson's variant of the Yahalom protocol



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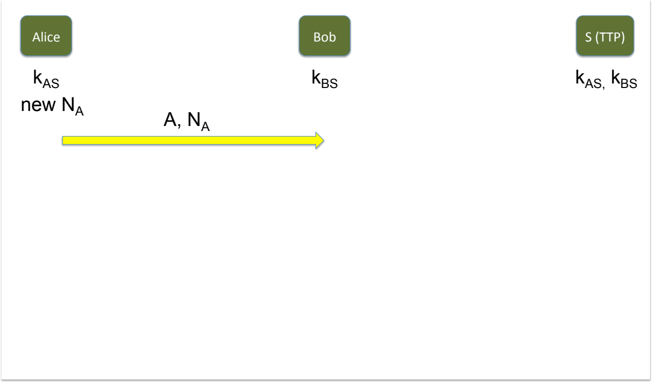

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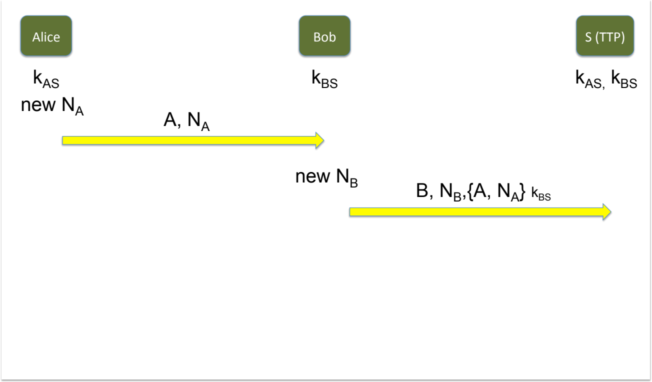

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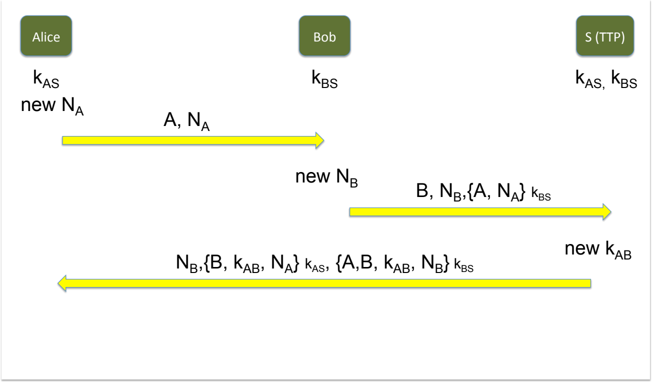

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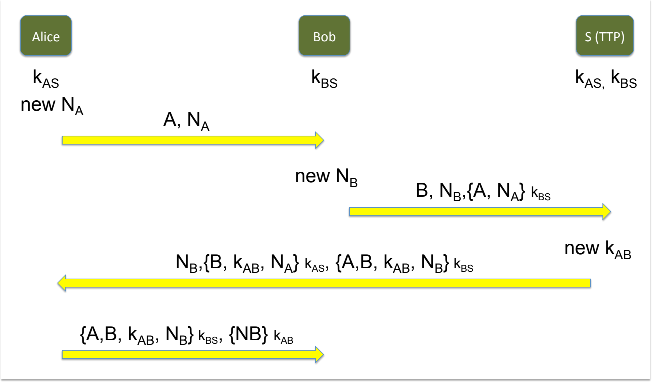

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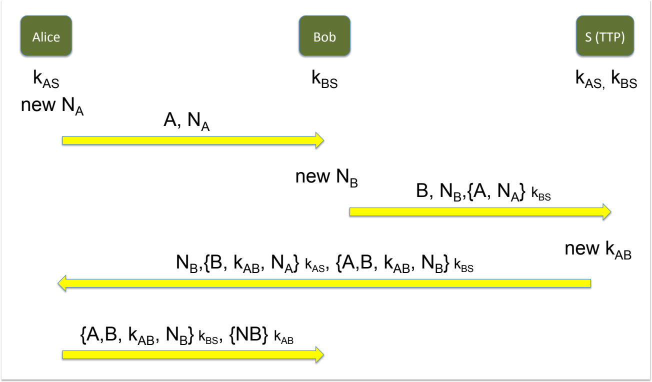
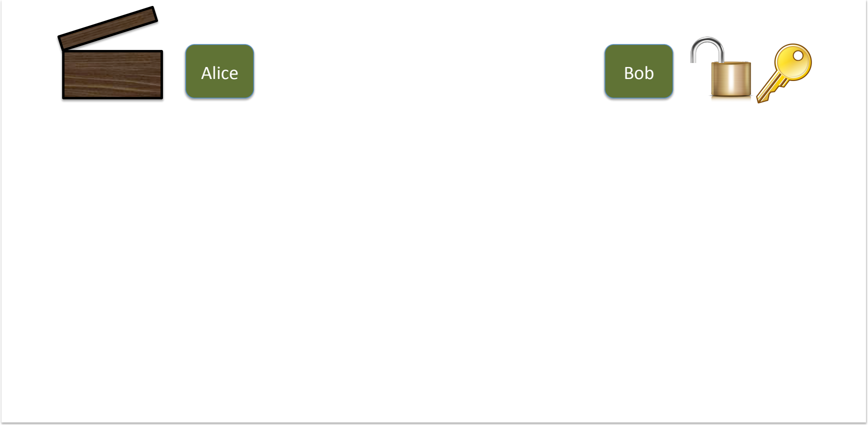


Question: can we establish a shared secret key without a TTP?
Answer: Yes!

Public-key encryption in pictures



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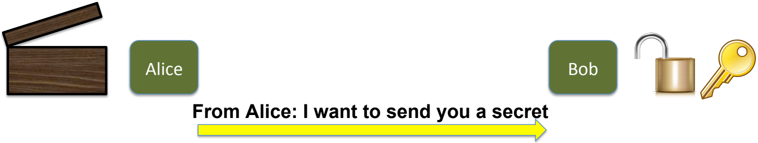

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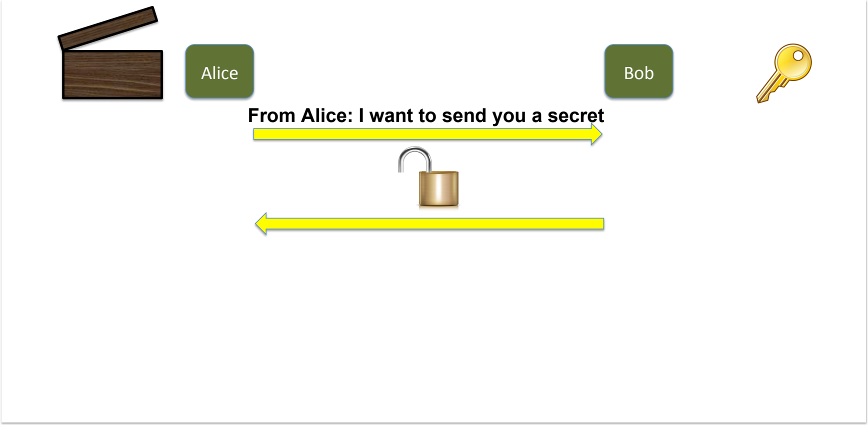

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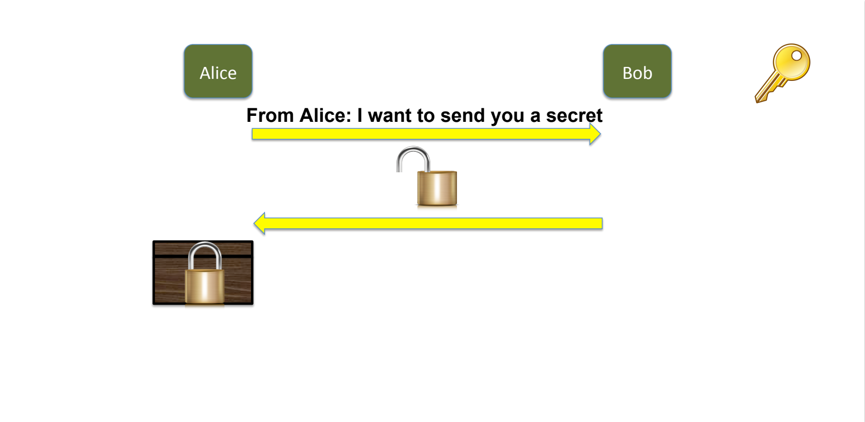

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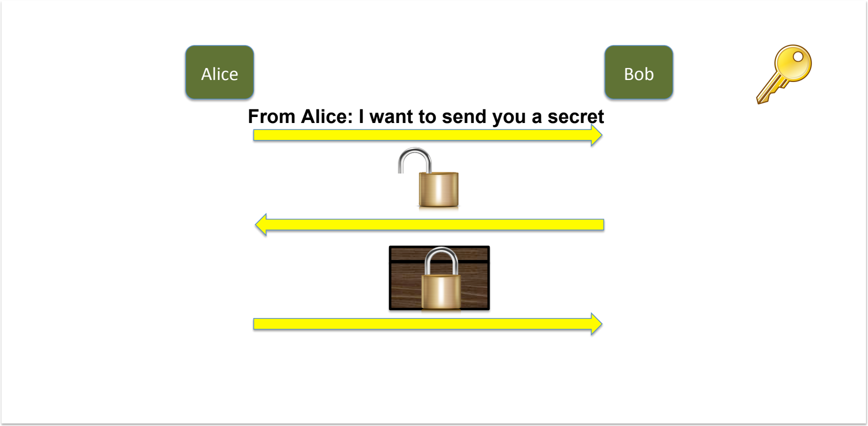

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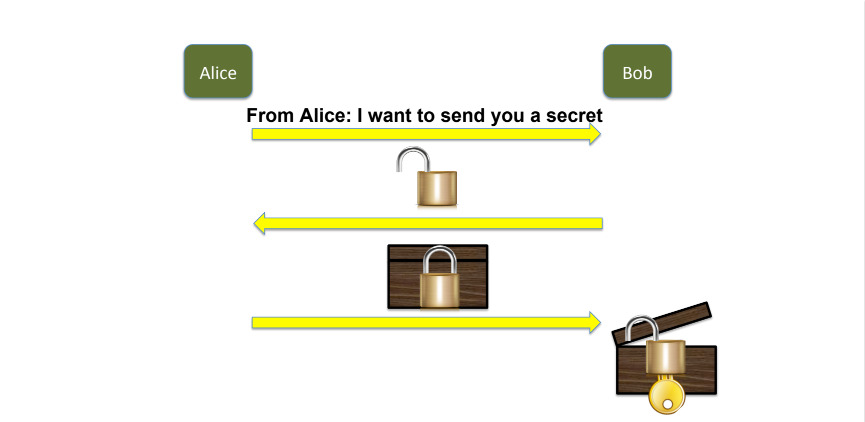
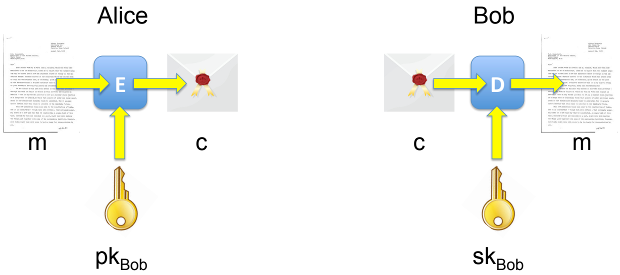


Public-key encryption

- key generation algorithm: $G : \to \mathcal{K} \times \mathcal{K}$
  encryption algorithm $E : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$
  decryption algorithm $D : \mathcal{K} \times \mathcal{C} \to \mathcal{M}$
  st. $\forall (sk, pk) \in G$, and $\forall m \in \mathcal{M}$, $D(sk, E(pk, m)) = m$



- the decryption key sk_{Bob} is secret (only known to Bob). The encryption key pk_{Bob} is known to everyone. And $sk_{Bob} \neq pk_{Bob}$

We need a bit of number theory now

Primes

Definition

$p \in \mathbb{N}$ is a **prime** if its only divisors are 1 and p

Ex: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Theorem

Every $n \in \mathbb{N}$ has a **unique factorization** *as a product of prime numbers (which are called its factors)*

Ex: $23244 = 2 \times 2 \times 3 \times 13 \times 149$

Relative primes

Definition

a and *b* in $\mathbb{Z}$ are **relative primes** if they have no common factors

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The Euler function $\phi(n)$ is the number of elements that are relative primes with n:

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- For p and q primes: $\phi(p \cdot q) = (p\text{-}1)(q\text{-}1)$

$\mathbb{Z}_n$

- Let $n \in \mathbb{N}$. We define $\mathbb{Z}_n = \{0, \ldots n\text{-}1\}$

$$\forall a \in \mathbb{Z},\ \forall b \in \mathbb{Z}_n,\ a \equiv b \pmod{n} \ \Leftrightarrow\ \exists k \in \mathbb{N}.\ a = b + k \cdot n$$

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Theorem

Let $n \in \mathbb{N}$. Let $x \in \mathbb{Z}_n$. x has a inverse in $\mathbb{Z}_n$ iff $\gcd(x, n) = 1$

$(\mathbb{Z}_N)^*$

- Let $n \in \mathbb{N}$. We define $(Z_n)^* = \{x \in \mathbb{Z}_n \mid \gcd(x, n) = 1\}$
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Theorem (Euler)

$\forall p$ prime, $(\mathbb{Z}_p)^*$ is a cyclic group, i.e.

$$\exists g \in (\mathbb{Z}_p)^*, \ \{g, g^2, g^3, \ldots, g^{p-2}\} = (\mathbb{Z}_p)^*$$

Intractable problems

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  input: $n \in \mathbb{N}$
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- DHP:
  input: prime p, generator g of $(\mathbb{Z}_p)^*$, $g^a \pmod{p}$, $g^b \pmod{p}$
  output: $g^{ab} \pmod{p}$

We can now go back and see how to establish a key without a TTP

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- Assumption: the DHP is hard in $(\mathbb{Z}_p)^*$
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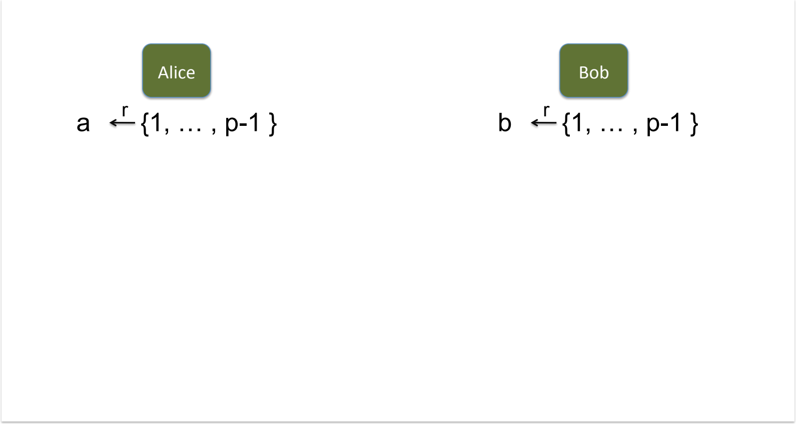

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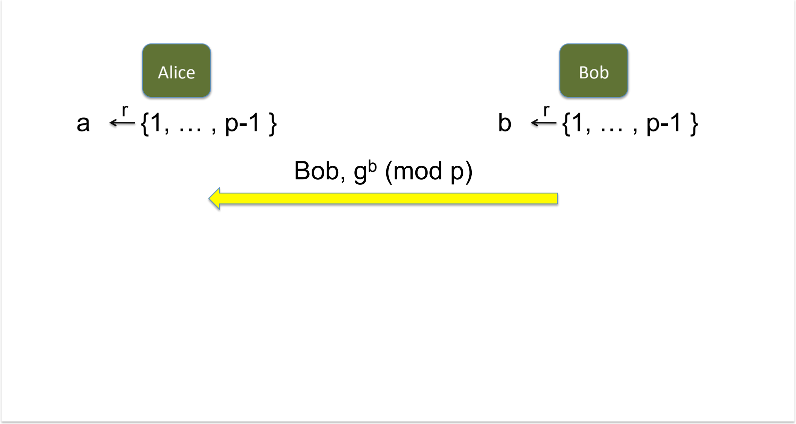

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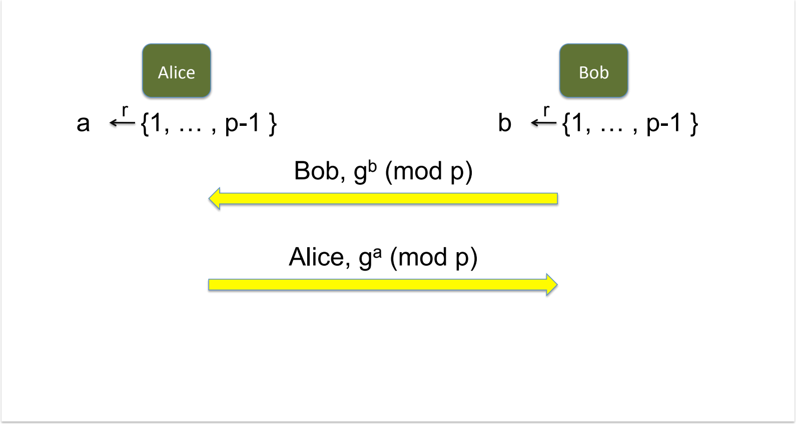

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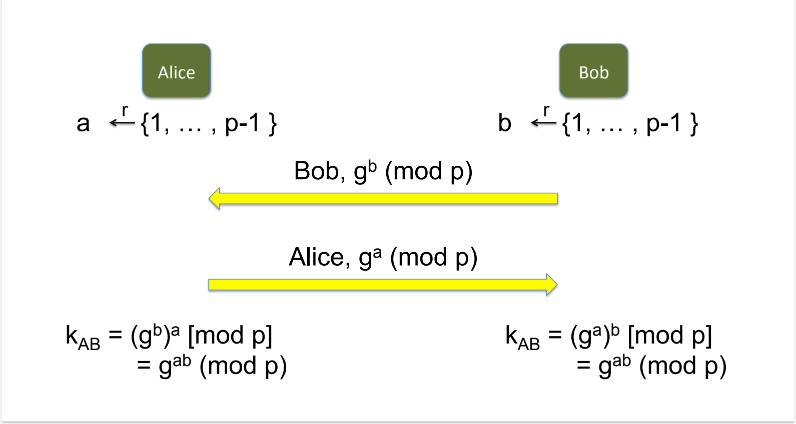

Man in the middle attack on DH

Alice

$a \xleftarrow{r} \{1, \dots, p-1\}$

Attacker

$a' \xleftarrow{r} \{1, \dots, p-1\}$

$b' \xleftarrow{r} \{1, \dots, p-1\}$

Bob

$b \xleftarrow{r} \{1, \dots, p-1\}$

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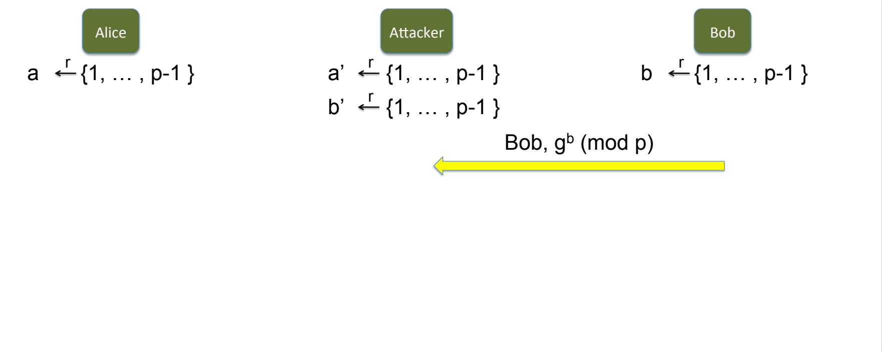

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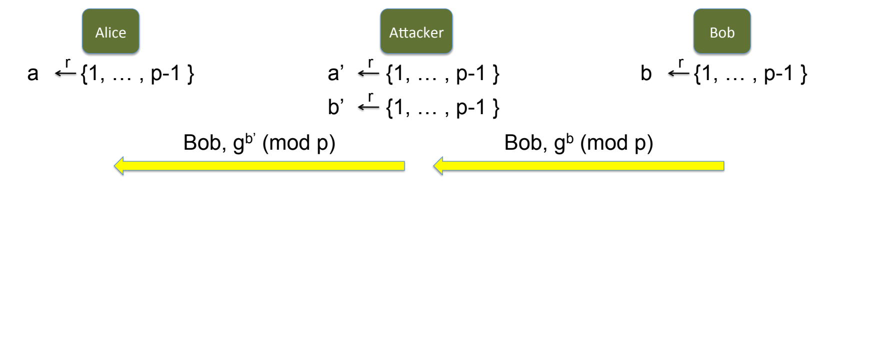

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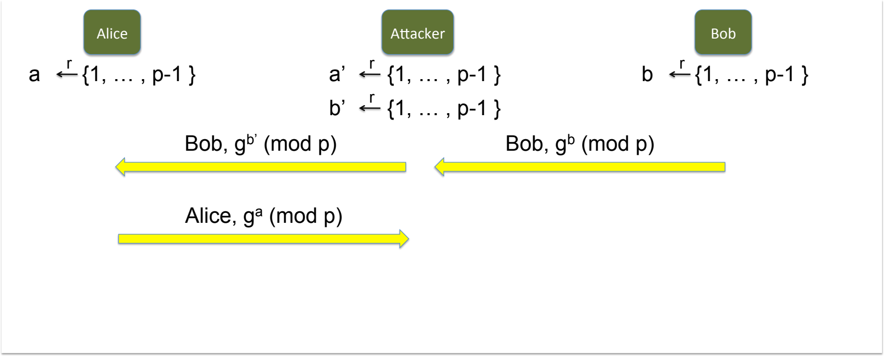

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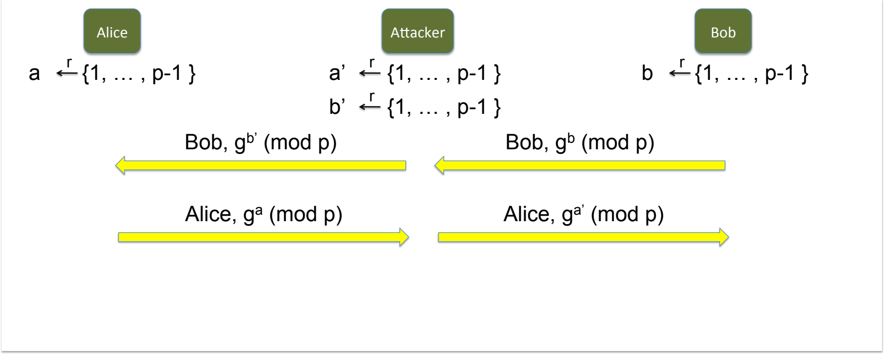

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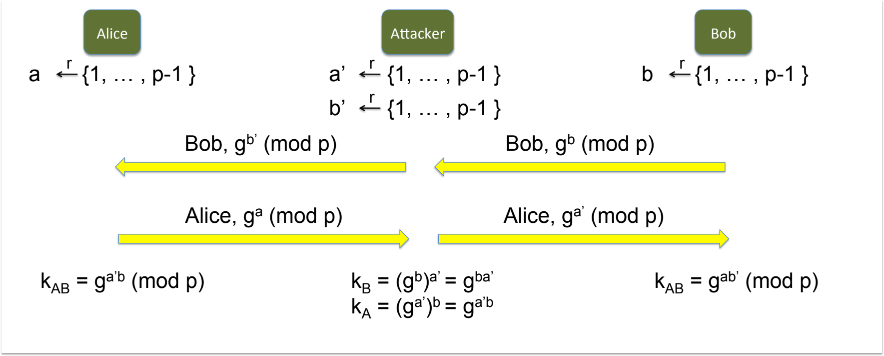

RSA trapdoor permutation

- $G_{RSA}() = (pk, sk)$ where $pk = (N, e)$ and $sk = (N, d)$ and $N = p \cdot q$ with p, q random primes and $e, d \in \mathbb{Z}$ st. $e \cdot d \equiv 1 \pmod{\phi(N)}$

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$$RSA^{-1}(sk, RSA(pk, x)) = (x^e)^d \pmod{N}$$

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- $G_{RSA}() = (pk, sk)$ where $pk = (N, e)$ and $sk = (N, d)$ and $N = p \cdot q$ with p, q random primes and $e, d \in \mathbb{Z}$ st. $e \cdot d \equiv 1 \pmod{\phi(N)}$
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How NOT to use *RSA*

(G_{RSA}, RSA, RSA^{-1}) is called raw *RSA*. Do not use raw *RSA*directly as an asymmetric cipher!

RSA is deterministic $\Rightarrow$ not secure against chosen plaintext attacks

(Details on the board)

ISO standard

Goal: build a CPA secure asymmetric cipher using (G_{RSA}, RSA, RSA^{-1})

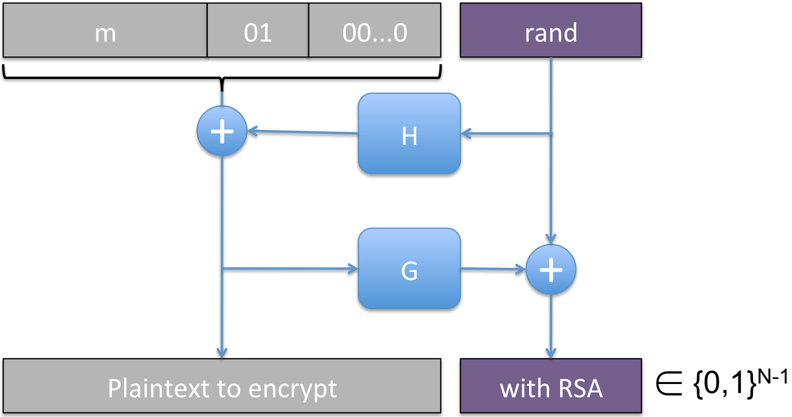
Let (E_s, D_s) be a symmetric encryption scheme over $(\mathcal{M}, \mathcal{C}, \mathcal{K})$
Let $H: (\mathbb{Z}_N)^* \to \mathcal{K}$

Build $(G_{RSA}, E_{RSA}, D_{RSA})$ as follows

- $G_{RSA}()$ as described above
- $E_{RSA}(pk, m)$:
  - pick random $x \in (\mathbb{Z}_N)^*$
  - $y \leftarrow RSA(pk, x)(= x^e)$
  - $k \leftarrow H(x)$
  - $E_{RSA}(pk, m) = y || E_s(k, m)$
- $D_{RSA}(pk, y||c) = D_s(H(RSA^{-1}(sk, y)), c)$

PKCS1 v2.0: *RSA*-OAEP

Goal: build a CPA secure asymmetric cipher using (G_{RSA}, RSA, RSA^{-1})



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$$D_{EG}(sk, E_{EG}(pk, x)) \;=\; (g^r)^{-d} \cdot m \cdot (g^d)^r \pmod{p}$$

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