Cryptography IV: Asymmetric Ciphers

Computer Security Lecture 5

David Aspinall

School of Informatics
University of Edinburgh

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Outline

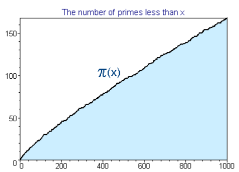
History of Public-Key Cryptography

- Idea attributed to Diffie and Hellman in 1975, but later revealed in British classified work of James Ellis in 1971 at GCHQ.
- Basic idea: alter traditional symmetry of cryptographic protocols to convey additional info in a *public key*. The sender uses the public key to convey a secret message to the recipient, without requiring a secure channel to share key information.
- Originally presented as a means of encrypting messages. In practice, public key algorithms are used to exchange symmetric keys
  - Public keys are *key encrypting keys*
  - Symmetric keys are *data encrypting keys*
- Public keys are also used to provide integrity through *digital signatures* (later lecture)

Prime numbers

- A natural number $p \geq 2$ is *prime* if 1 and p are its only positive divisors.
- For $x \geq 17$, then $\pi(x)$, the number of primes less than or equal to x, is approximated by:

$$\frac{x}{\ln x} < \pi(x) < 1.25506 \frac{x}{\ln x}$$



Fundamental theorem of arithmetic

Every natural number $n \geq 2$ has a unique factorization as a product of prime powers: $p_1^{e_1} \cdots p_n^{e_n}$ for distinct primes p_i and positive e_i.

Relative primes

- Two integers a and b are **relatively prime** if $\gcd(a, b) = 1$, i.e., a and b have no common factors.
- The *Euler totient function* $\phi(n)$ is the number of elements of $\{1, \ldots, n-1\}$ relatively prime to n.
- Given the factorisation of n, it's easy to compute $\phi(n)$.
  - For prime n, $\phi(n) = n - 1$
  - For distinct primes p, q, $\phi(pq) = (p-1)(q-1)$.
- An integer n is said to be ***B*-smooth** wrt a positive bound B, if all its prime factors are $\leq B$.
  - We say just "n is smooth" when $B \ll n$.
  - Smooth numbers are easy to factor, leading to efficient factoring algorithms for large composites containing primes p with for which $p - 1$ is smooth (**Pollard's $P - 1$ method**).

Integers modulo n: $\mathbf{Z}_n$ and $\mathbf{Z}_n^*$

- Let n be a positive integer. The set

$$\mathbf{Z}_n = \{0, \ldots, n-1\}$$

  contains (equivalence classes of) integers mod n.
- Let $a \in \mathbf{Z}_n$. The **multiplicative inverse** of a modulo n is the unique $x \in \mathbf{Z}_n$ such that

$$ax \equiv 1 \pmod{n}.$$

  Such an x exists iff $\gcd(a, n) = 1$.
- We can define a *multiplicative group* $\mathbf{Z}_n^*$ by

$$\mathbf{Z}_n^* = \{a \in \mathbf{Z}_n \mid \gcd(a, n) = 1\}.$$

- Facts:
  - $\mathbf{Z}_n^*$ is closed under multiplication
  - $|\mathbf{Z}_n^*| = \phi(n)$
  - For prime n, $\mathbf{Z}_n^* = \{1, \ldots, n-1\}$.

Properties of integers in $\mathbf{Z}_n^*$

Fermat's little theorem

If p is prime and $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$.

Euler's theorem

If $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$.

- Fermat's little theorem is used in several places, e.g. a simple probabilistic primality test:
  - repeatedly test $a^{p-1} \mod p$ for random a
  - Miller-Rabin improves this (*Carmichael* numbers fail)
- Euler's theorem allows reduction of large powers.
  - $5^{79} \mod 6 = (5^2 * 5^2)^{19} * 5^3 = 1^{19} * 125 \mod 6 = 5$
  - Generally: if $x \equiv y \pmod{\phi(n)}$, then $a^x \equiv a^y \pmod{n}$.

Cyclic groups

- Let $a \in \mathbf{Z}_n^*$.
  The *order* of a is the least $t > 0$ st $a^t \equiv 1 \pmod{n}$.
- If $g \geq 2$ has order $\phi(n)$, then $\mathbf{Z}_n^*$ is **cyclic** and g is a **generator** (aka *primitive root*) of $\mathbf{Z}_n^*$.
- $\mathbf{Z}_n^*$ is cyclic iff $n = 2, 4, p^k, 2p^k$ for odd primes p.
- The **discrete logarithm** of b wrt g is the x st $g^x \equiv b \pmod{n}$.
- There is an efficient algorithm for computing discrete logs in $\mathbf{Z}_p^*$ if $p-1$ has smooth factors.

Example: $\mathbf{Z}_5^*$

- Here is the multiplication table for $\mathbf{Z}_5^*$, showing $xy \pmod{5}$.

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

  - $|\mathbf{Z}_5^*| = \phi(5) = 4$
  - Inverses: $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$.
  - Notice $2^4 = 2 * 2 * 2 * 2 = 1$, also $3^4 = 4^4 = 1$.
  - Generators are: 2, 3, 4.
  - In $\mathbf{Z}_5^*$, the discrete log of 4 for base 3 is 2

Example: $\mathbf{Z}_{15}^*$

- Here is the multiplication table for $\mathbf{Z}_{15}^*$, showing $xy \pmod{15}$.

	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

  - $|\mathbf{Z}_{15}^*| = \phi(15) = (3-1) * (5-1) = 8$.
  - This group is not cyclic.
    Exercise: find orders of each element.

RSA

- A key-pair is based on product of two large, distinct, random secret primes, $n{=}pq$ with p and q roughly the same size, together with a random integer e with $1 < e < \phi$ and $\gcd(e, \phi) = 1$, where

$$\phi = \phi(n) = (p-1)(q-1).$$

  Public key is (n, e) and n is called the *modulus*.
- Private key is d, unique s.t. $ed \equiv 1 \pmod{\phi}$.
- Message and cipher space $\mathcal{M} = \mathcal{C} = \{0, \ldots, n-1\}$.
- Encryption is exponentiation with public key e.
  Decryption is exponentiation with private key d.

$$\begin{aligned} E_{(n,e)}(m) &= m^e \bmod n \\ D_d(c) &= c^d \bmod n \end{aligned}$$

- Decryption works, since for some k, $ed = 1 + k\phi$ and

$$(m^e)^d \equiv m^{ed} \equiv m^{1+k\phi} \equiv mm^{k\phi} \equiv m \pmod{n}$$

  using Fermat's theorem. (**Exercise:** fill details in).

RSA remarks

- Recall that RSA is an example of a **reversible** public-key encryption scheme. This is because e and d are symmetric in the definition. RSA digital signatures make use of this.
- RSA is often used with randomization (e.g., **salting** with random appendix) to prevent chosen-plaintext and other attacks.
- It's the most popular and cryptanalysed public-key algorithm. Largest modulus factored in the (now defunct) RSA challenge is 768 bits (232 digits), factored using the Number Field Sieve (NFS) on 12 December 2009.
  - It took the equivalent of 2000 years of computing on a single core 2.2GHz AMD Opteron. On the order of 2^{67} instructions were carried out.
  - Factoring a 1024 bit modulus would take about 1000 times more work (soon achievable).

RSA usage and key size

- In practice, RSA is used to encrypt symmetric keys, not messages.
- Like most public key algorithms, the RSA key size is larger, and the computations are more expensive than symmetric schemes such as AES.
- This is believed to be a necessary result of the key being publicly available.
- Consider the complexity of brute force attacks for an n-bit key:
  - A worst-case attack algorithm on a symmetric cipher would take $O(2^n)$ effort (*exponential*).
  - A worst-case attack algorithm for RSA is dependent upon the complexity of factoring, and thus would take $O(e^{o(n)})$ (*sub-exponential*)
  - (unless you have a quantum computer)

Cryptographic Reference Problems I

FACTORING Integer factorization. Given positive n, find its prime factorization, i.e., distinct p_i such that $n = p_1^{e_1} \cdots p_n^{e_n}$ for some $e_i \geq 1$.

SQRROOT Given a such that $a \equiv x^2 \pmod{n}$, find x.

RSAP RSA inversion. Given n such that $n = pq$ for some odd primes $p \neq q$, and e such that $\gcd(e, (p-1), (q-1)) = 1$, and c, find m such that $m^e \equiv c \pmod{n}$.

Note: SQRROOT $=_P$ FACTORING and RSAP $\leq_P$ FACTORING

- $A \leq_P B$ means there is a polynomial time (efficient) reduction from problem A to problem B.
- $A =_P B$ means $A \leq_P B$ and $B \leq_P A$
- So: RSAP is no harder than FACTORING.
  Is it easier? An open question.

Cryptographic Reference Problems II

DLP Discrete logarithm problem. Given prime p, a generator g of $\mathbf{Z}_p^*$, and an element $a \in \mathbf{Z}_p^*$, find the integer x, with $0 \leq x \leq p-2$ such that $g^x \equiv a \pmod{p}$.

DHP Diffie-Hellman problem. Given prime p, a generator g of $\mathbf{Z}_p^*$, and elements $g^a \bmod p$ and $g^b \bmod p$, find $g^{ab} \bmod p$.

Note: DHP$\leq_P$DLP.
In some cases, DHP$=_P$DLP.

Diffie-Hellman key agreement

- Diffie-Hellman key agreement allows two principals to agree a shared key without authentication. Initial setup: choose and publish a large "secure" prime p and generator g of $\mathbf{Z}_p^*$.

  Message 1. $A \to B$: $g^x \bmod p$
  Message 2. $B \to A$: $g^y \bmod p$

  - A chooses random x, $1 \leq x < p-1$, sends msg 1.
  - B chooses random y, $1 \leq y < p-1$, sends msg 2.
  - B receives g^x, computes shared key $K = (g^x)^y \bmod p$.
  - A receives g^y, computes shared key $K = (g^y)^x \bmod p$.
- Security rests on intractability of DHP for p and g. Protocol is safe against passive adversaries, but not active ones.
  Exercise: try some artificial examples with $p = 11$, $g = 2$. Show a MITM attack against the protocol.

ElGamal encryption

- A key-pair is based on a large random prime p and generator g of $\mathbf{Z}_p^*$, and a random integer d. Public key: $(p, g, g^d \bmod p)$, private key: d.
- The message space $\mathcal{M} = \{0, \ldots, p-1\}$, and the encryption operation is given by selecting a random integer r and computing a pair:

$$E_{(p,g,g^d)}(m) = (e, c) \quad \text{where} \quad e = g^r \bmod p, \quad c = m(g^d)^r \bmod p.$$

- Decryption takes an element of ciphertext $\mathcal{C} = \mathcal{M} \times \mathcal{M}$, and computes:

$$D_d(e, c) = e^{-d} c \bmod p \qquad \text{where } e^{-d} = e^{p-1-d} \bmod p.$$

- Decryption works because $e^{-d} = g^{-dr}$, so

$$D_d(e, c) \equiv g^{-dr} m g^{dr} \equiv m \pmod{p}.$$

- This is like using Diffie-Hellman to agree a key g^{dr} and encrypting m by multiplication.

ElGamal remarks

- ElGamal is an example of a **randomized** encryption scheme, so no need to add salt. Security relies in intractability of DHP. Choosing different r for different messages is critical. **Exercise:** why?
- Efficiency:
  - ciphertext twice as long as plaintext
  - encryption requires two modular exponentiations, which can be sped up by picking the random r with some additional structure (with care).
- The prime p and generator g can be fixed for the system, reducing the size of public keys. Then exponentiation can be speeded up by precomputation; however, so can the best-known algorithm for calculating discrete logarithms, so a larger modulus would be warranted.
- The **security** of ElGamal encryption and signing is based on the intractability of the DHP for p. Several other conditions are required.

Summary: Current Public Key algorithms

- **RSA**, **ElGamal** already described.
- **Elliptic curve** schemes. Use ElGamal techniques. Have shorter keys for same amount of security.
- **Rabin** encryption. Based on SQRROOT problem.
- Probabilistic schemes, which achieve **provable security** based on Random Oracle Method (ROM) arguments.
- **Cramer-Shoup**. Extends ElGamal with use of hash functions in critical places to provide provable security without ROM. Less efficient than ElGamal: slower and ciphertext twice as long.

References

- Alfred J. Menezes, Paul C. Van Oorschot, and Scott A. Vanstone, editors. *Handbook of Applied Cryptography*. CRC Press Series on Discrete Mathematics and Its Applications. CRC Press, 1997. Online version at `http://www.cacr.math.uwaterloo.ca/hac`.
- Nigel Smart. *Cryptography: An Introduction*. McGraw-Hill, 2003. Third edition online: `http://www.cs.bris.ac.uk/~nigel/Crypto_Book/`

Recommended Reading

Chapter **11**, 12, 13 of Smart (3rd Ed).