Cryptography IV: Asymmetric Ciphers

Computer Security Lecture 7

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Outline

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Background

RSA

Diffie-Hellman

ElGamal

Summary

History

- Originally attributed to Diffie and Hellman in 1975, but later revealed in British classified work of James Ellis in 1971
- Basic idea: alter traditional symmetry of cryptographic protocols to convey additional info in a *public key*. The sender uses the public key to convey a secret message to the recipient, without requiring a secure channel to share key information.
- Originally presented as a means of encrypting messages. In practice, public key algorithms are used to exchange symmetric keys
  - Public keys are *key encrypting keys*
  - Symmetric keys are *data encrypting keys*
- Public keys are also used to provide integrity through *digital signatures* (later lecture)

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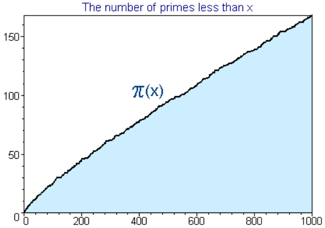

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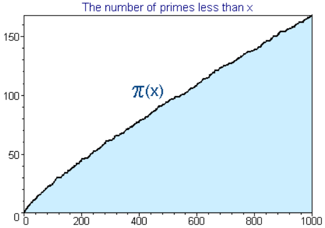

Fundamental theorem of arithmetic

Every natural number $n \geq 2$ has a unique factorization as a product of prime powers: $p_1^{e_1} \cdots p_n^{e_n}$ for distinct primes p_i and positive e_i.

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  - We say just "n is smooth" when $B \ll n$.
  - Smooth numbers are easy to factor, leading to efficient factoring algorithms for large composites containing primes p with for which $p-1$ is smooth (Pollard's $P-1$ **method**).

Integers modulo n: $\mathbf{Z}_n$ and $\mathbf{Z}_n^*$

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  - $5^{79} \mod 6 = (5^2 * 5^2)^{19} * 5^3 = 1^{19} * 125 \mod 6 = 5$
  - Generally: if $x \equiv y \pmod{\phi(n)}$, then $a^x \equiv a^y \pmod{n}$.

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- There is an efficient algorithm for computing discrete logs in $\mathbf{Z}_p^*$ if $p - 1$ has smooth factors.

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- Here is the multiplication table for $\mathbf{Z}_5^*$, showing $xy \pmod 5$.

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Example: $\mathbf{Z}_{15}^*$

- Here is the multiplication table for $\mathbf{Z}_{15}^*$, showing xy (mod 15).

	1	2	4	7	8	11	13	14
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4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

- $|\mathbf{Z}_{15}^*| = \phi(15) = (3-1)*(5-1) = 8.$
- This group is not cyclic.
  Exercise: find orders of each element.

Outline

RSA

- A key-pair is based on product of two large, distinct, random secret primes, $n = pq$ with p and q roughly the same size, together with a random integer e with $1 < e < \phi$ and $\gcd(e, \phi) = 1$, where

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- Encryption is exponentiation with public key e.
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  $$\begin{aligned} E_{(n,e)}(m) &= m^e \bmod n \\ D_d(c) &= c^d \bmod n \end{aligned}$$

- Decryption works, since for some k, $ed = 1 + k\phi$ and

  $$(m^e)^d \equiv m^{ed} \equiv m^{1+k\phi} \equiv m m^{k\phi} \equiv m \pmod{n}$$

  using Fermat's theorem. (**Exercise:** fill details in).

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- RSA is often used with randomization (e.g., **salting** with random appendix) to prevent chosen-plaintext and other attacks.
- It's the most popular and cryptanalysed public-key algorithm. Largest modulus factored in the (now defunct) RSA challenge is 768 bits (232 digits), factored using the Number Field Sieve (NFS) on 12 December 2009.

RSA remarks

- Recall that RSA is an example of a **reversible** public-key encryption scheme. This is because *e* and *d* are symmetric in the definition. RSA digital signatures make use of this.
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- It's the most popular and cryptanalysed public-key algorithm. Largest modulus factored in the (now defunct) RSA challenge is 768 bits (232 digits), factored using the Number Field Sieve (NFS) on 12 December 2009.
    - It took the equivalent of 2000 years of computing on a single core 2.2GHz AMD Opteron. On the order of 2^{67} instructions were carried out.
    - Factoring a 1024 bit modulus would take about 1000 times more work (and would be achievable in less than 5 years from now).

RSA Remarks . . .

- In practice, RSA is used to encrypt symmetric keys, not messages
- Like most public key algorithms, the RSA key size is larger, and the computations are more expensive (compared to AES, for example)
- This is believed to be a necessary result of the key being publicly available
- With regard to attack complexity based upon an n-bit key
  - A worst-case attack algorithm on a symmetric cipher would take $O(2^n)$ work (*exponential*).
  - A worst-case attack algorithm for RSA is dependent upon the complexity of factoring, and thus would take $O(e^{o(n)})$ (*sub-exponential*)

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FACTORING Integer factorization. Given positive n, find its prime factorization, i.e., distinct p_i such that $n = p_1^{e_1} \cdots p_n^{e_n}$ for some $e_i \geq 1$.

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Note: SQRROOT $=_P$ FACTORING and RSAP $\leq_P$ FACTORING

- $A \leq_P B$ means there is a polynomial time (efficient) reduction from problem A to problem B.
- $A =_P B$ means $A \leq_P B$ and $B \leq_P A$
- So: RSAP is no harder than FACTORING.
  Is it easier? An open question.

Outline

Cryptographic Reference Problems II

DLP Discrete logarithm problem. Given prime p, a generator g of $\mathbf{Z}_p^*$, and an element $a \in \mathbf{Z}_p^*$, find the integer x, with $0 \leq x \leq p-2$ such that $g^x \equiv a \pmod{p}$.

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In some cases, DHP$=_P$DLP.

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  - A receives g^y, computes shared key $K = (g^y)^x \bmod p$.
- Security rests on intractability of DHP for p and g. Protocol is safe against passive adversaries, but not active ones.
  Exercise: try some artificial examples with $p = 11$, $g = 2$. Show a MITM attack against the protocol.

Outline

- Background
- RSA
- Diffie-Hellman
- ElGamal
- Summary

ElGamal encryption

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- This is like using Diffie-Hellman to agree a key g^{dr} and encrypting m by multiplication.

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- The **security** of ElGamal encryption and signing is based on the intractability of the DHP for p. Several other conditions are required.

Outline

Summary: Current Public Key algorithms

- **RSA**, **ElGamal** already described.
- **Elliptic curve** schemes. Use ElGamal techniques. Have shorter keys for same amount of security.
- **Rabin** encryption. Based on SQRROOT problem.
- Probabilistic schemes, which achieve **provable security** based on Random Oracle Method (ROM) arguments.
- **Cramer-Shoup**. Extends ElGamal with use of hash functions in critical places to provide provable security without ROM. Less efficient than ElGamal: slower and ciphertext twice as long.

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  CRC Press Series on Discrete Mathematics and Its Applications. CRC Press, 1997.
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  http://www.cacr.math.uwaterloo.ca/hac.
- Nigel Smart. *Cryptography: An Introduction*.
  McGraw-Hill, 2003. Third edition online: http://www.cs.bris.ac.uk/~nigel/Crypto_Book/

Recommended Reading

Chapter **11**, 12, 13 of Smart (3rd Ed).