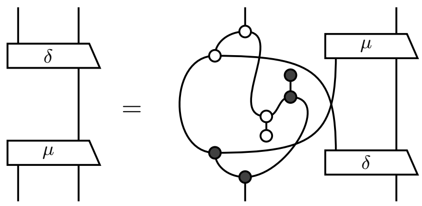
Categories and Quantum Informatics
mock exam

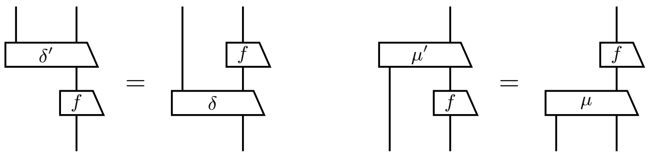
Question 1

In a symmetric monoidal dagger category $\mathbf{C}$, suppose that $(M, ,)$ is a commutative dagger-Frobenius monoid, and $(M, ,)$ is a dagger-Frobenius monoid, which together are strongly complementary. Then we can build a new monoidal category $\mathbf{X_C}$ in the following way:

- **Objects** are triples (A, μ, δ), where A is an object of $\mathbf{C}$, carrying a dagger-comodule structure $A \xrightarrow{\delta} M \otimes A$ for $(M, ,)$, and a dagger-module structure $M \otimes A \xrightarrow{\mu} A$ for $(M, ,)$, satisfying the following compatibility equation:

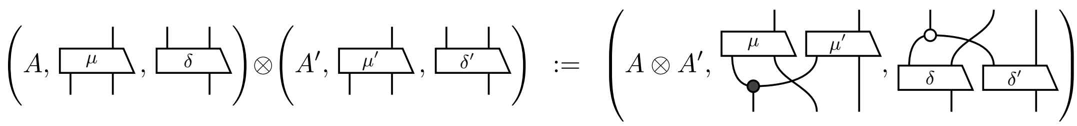


- **Morphisms** $(A, \mu, \delta) \to (A', \mu', \delta')$ are morphisms $A \xrightarrow{f} A'$ in $\mathbf{C}$, satisfying:



  Composition is as in $\mathbf{C}$.

- **Tensor product** of morphisms is inherited from $\mathbf{C}$, and of objects is defined as:



This construction is important in some models of topological quantum computation.

(a)

The description of the monoidal structure given above is not complete. Find a unit object I and natural isomorphisms α, λ and ρ that make $\mathbf{X_C}$ into a monoidal category.

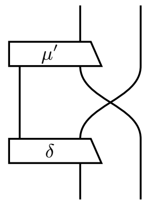
(b)

Show that $\mathbf{X_C}$ is a monoidal dagger category, with the dagger of a morphism given by its dagger as a morphism in $\mathbf{C}$.

(c)

For any two objects (A, μ, δ) and (A', μ', δ') of $\mathbf{X_C}$, define a morphism in $\mathbf{C}$ as follows:

$\sigma_{(A,\mu,\delta),(A',\mu',\delta')} \quad :=$



Show that this is unitary.

(d)

Show that for any two objects (A, μ, δ), (A', μ', δ') of $\mathbf{X_C}$, the morphism $\sigma_{(A,\mu,\delta),(A',\mu',\delta')}$ gives a morphism in $\mathbf{X_C}$ of type $(A, \mu, \delta) \otimes (A', \mu', \delta') \to (A', \mu', \delta') \otimes (A, \mu, \delta)$.

(e)

Show that the morphisms $\sigma_{(A,\mu,\delta),(A',\mu',\delta')}$ make $\mathbf{X_C}$ into a braided monoidal dagger category.

(f)

Let $\mathbf{C} = \mathbf{Rel}$. Take $M = \mathbb{Z}_2 = \{0, 1\}$, and define $\bullet$ and $\circ$ as follows, where '+' is addition modulo 2, for all $x, y \in M$:

$$\bullet\colon (x, x) \sim x$$
$$\circ\colon (x, y) \sim x + y.$$

Show that this is a strongly complementary pair. Take $A = \mathbb{Z}_2$, and define $\mu : M \otimes A \to A$ and $\delta : A \to M \otimes A$ as follows, for all $a \in A$:

$$\mu\colon (x, a) \sim x + a$$
$$\delta\colon a \sim (1, a)$$

Show that (A, μ, δ) is an object of $\mathbf{X_{Rel}}$. Compute $\sigma_{(A,\mu,\delta),(A,\mu,\delta)}$ explicitly, and hence show that $\mathbf{X_{Rel}}$ is not symmetric monoidal.

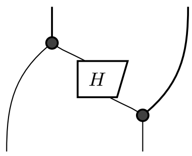
Question 2

This question is about using complementary structures to model Shor's quantum error correction protocol. Let A be an object in a dagger compact category, carrying a classical structure written . Let $H : A \to A$ be a unitary morphism such that the following morphisms define complementary classical structures, up to a scalar factor:

(1)

(a)

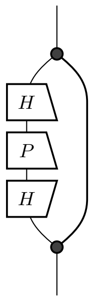
Show that the following composite is unitary, up to a scalar factor:



(2)

(b)

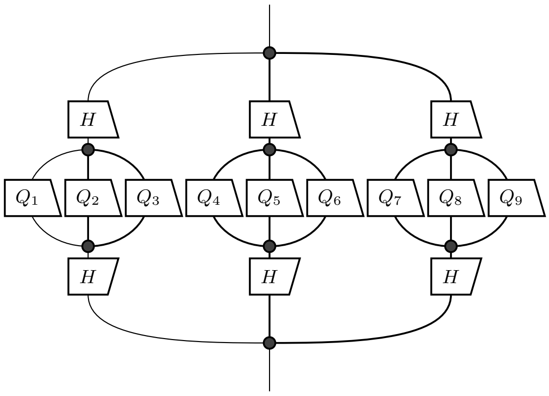
Supposing that $P : A \to A$ satisfies $P \circ {} = {} \circ (\mathrm{id}_A \otimes P) = {} \circ (P \otimes \mathrm{id}_A)$, show that the following is equal to the identity, up to a scalar factor:



(3)

(c)

Let $Q_i : A \to A$ for $1 \leq i \leq 9$ be a family of morphisms, such that Q_i equals the identity for at least 7 values of i. Show that the following is equal to the identity, up to a scalar factor:



(4)