Categories and Quantum Informatics: Coursework

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Produce a coherent essay of about 5 pages, grounded in concrete mathematical proofs, about the use of monads in programming, where you touch at least on the following points. You can use any external material you like, but identify your sources, and keep your essay self-contained.

A monad on a category \( \mathbf{C} \) consists of a functor \( F : \mathbf{C} \to \mathbf{C} \) together with two natural transformations \( \eta : A \to F(A) \) and \( \mu : F(F(A)) \to A \) making the following diagrams commute:

\[
\begin{array}{c}
F(A) \xrightarrow{F(\eta)} F^2(A) \xrightarrow{F(\mu)} F^3(A) \\
\downarrow \eta \quad \quad \quad \quad \quad \quad \downarrow \mu \\
F^2(A) \xrightarrow{\mu} F(A) \xleftarrow{\mu} F^2(A)
\end{array}
\]

1. Given a monad \( (F, \mu, \eta) \) on \( \mathbf{C} \), show how to make a new category \( \mathbf{C}_F \) with the same objects as \( \mathbf{C} \), but whose morphisms \( A \to B \) are morphisms \( A \to F(B) \) in \( \mathbf{C} \). Find a functor \( \mathbf{C} \to \mathbf{C}_F \).

2. Show how the following are monads on \( \mathbf{Set} \):
   - exception: \( F(A) = A + 1 \) (where + is disjoint union and 1 is a fixed singleton set)
   - log: \( F(A) = A \times X \) for a fixed set \( X \)
   - input: \( F(X) = A^X \) for a fixed set \( X \) (where \( A^X \) is the set of functions \( X \to A \))

Discuss how we might think of morphisms in \( \mathbf{C} \) as computations without side-effects, and morphisms in \( \mathbf{C} \) as computations that may have side-effects modelled by \( F \), in each of these three cases.

3. If \( \mathbf{C} \) is monoidal, we might expect the monad \( F \) to cooperate with the monoidal structure. Look up and discuss the definition of a strong monad: it involves a natural transformation

\[
st_{A,B} : F(A) \otimes B \to F(A \otimes B).
\]

Show that every monad on \( (\mathbf{Set}, \times, 1) \) is strong. Discuss what it might mean for a monad on \( \mathbf{FHilb} \) to be strong in terms of quantum theory.

4. If \( \mathbf{C} \) is symmetric monoidal, strength induces a similar natural transformation \( A \otimes F(B) \to F(A \otimes B) \). Combining the two strength maps gives two natural transformations \( F(A) \otimes F(B) \to F(A \otimes B) \). The monad is called commutative when these two double strength maps coincide. Which of the above three monads are commutative on \( \mathbf{Set} \)? Discuss what it might mean for a monad on \( \mathbf{FHilb} \) to be commutative in terms of quantum theory.

5. Look up the notion of an adjunction. Discuss the relation between dual objects in a monoidal category, monads, and adjunctions, particularly with a view to regarding the objects and morphisms of \( \mathbf{C} \) as the types and functions of a programming language.