Categories and Quantum Informatics: Coursework

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Spring 2018

A *monad* on a category $\mathbf{C}$ consists of a functor $F\colon \mathbf{C} \to \mathbf{C}$ together with two natural transformations $\eta_A\colon A \to F(A)$ and $\mu_A\colon F(F(A)) \to F(A)$ making the following diagrams commute:

$$
\begin{tikzcd}
F(A) \arrow[r, "F(\eta_A)"] \arrow[d, "\eta_{F(A)}"'] \arrow[dr, "\mathrm{id}_{F(A)}"] & F^2(A) \arrow[d, "\mu_A"] & F^3(A) \arrow[l, "F(\mu_A)"'] \arrow[d, "\mu_{F(A)}"] \\
F^2(A) \arrow[r, "\mu_A"'] & F(A) & F^2(A) \arrow[l, "\mu_A"]
\end{tikzcd}
$$

1. Given a monad (F, μ, η) on $\mathbf{C}$, show how to make a new category $\mathbf{C}_F$ with the same objects as $\mathbf{C}$, but whose morphisms $A \to B$ are morphisms $A \to F(B)$ in $\mathbf{C}$. Find a functor $\mathbf{C} \to \mathbf{C}_F$.

2. Show how the following are monads on $\mathbf{Set}$:

   - exception: $F(A) = A + 1$ (where $+$ is disjoint union and 1 is a fixed singleton set)
   - log: $F(A) = A \times X$ for a fixed set X
   - input: $F(A) = A^X$ for a fixed set X (where A^X is the set of functions $X \to A$)
   - nondeterminism: $F(A) = \mathcal{P}(A)$, the powerset of A

   Discuss how we might think of morphisms in $\mathbf{C}$ as computations without side-effects, and morphisms in $\mathbf{C}$ as computations that may have side-effects modelled by F, in each of these four cases.

3. If $\mathbf{C}$ is monoidal, we might expect the monad F to cooperate with the monoidal structure. Look up and discuss the definition of a *strong* monad: it involves a natural transformation

   $$\mathrm{st}_{A,B}\colon F(A) \otimes B \to F(A \otimes B).$$

   Show that every monad on $(\mathbf{Set}, \times, 1)$ is strong.

4. If $\mathbf{C}$ is symmetric monoidal, strength induces a similar natural transformation $A \otimes F(B) \to F(A \otimes B)$. Combining the two strength maps gives two natural transformations $F(A) \otimes F(B) \to F(A \otimes B)$. The monad is called *commutative* when these two double strength maps coincide. Which of the above four monads are commutative on $\mathbf{Set}$?

5. Consider the nondeterminism monad F on $\mathbf{Set}$ as in part (2). Show that the category $\mathbf{Set}_F$ it induces as in part (1) is equivalent to $\mathbf{Rel}$.