Categories and Quantum Informatics

Week 8: Complete positivity

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THE UNIVERSITY of EDINBURGH informatics

Overview

- Completely positive maps:
  pure states/evolutions vs mixed ones
- Categories of completely positive maps:
  everything happily in one category
- Classical structures:
  operational view, broadcasting

Mixed states

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Partial trace is unique map $\mathrm{Tr}_K \colon \mathbf{Hilb}(H \otimes K, H \otimes K) \to \mathbf{Hilb}(H, H)$
satisfying $\mathrm{Tr}_K(\rho \otimes \sigma) = \mathrm{Tr}(\sigma) \cdot \rho$.

Partial trace of pure state can be mixed.

Mixed measurements

positive operator-valued measure (POVM) on a Hilbert space H is a family of positive maps $H \xrightarrow{f_i} H$ with $\sum_i f_i = \mathrm{id}_H$

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Born rule: for positive operator–valued measure $\{f_i\}$ and normalized density matrix $H \xrightarrow{\rho} H$, the *probability of outcome i* is $\langle\psi|f_i|\psi\rangle$.

Mixed states, categorically

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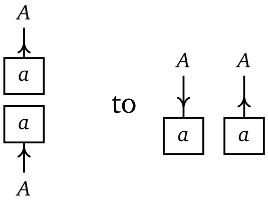
Step 1: consider $p = a \circ a^\dagger : A \to A$ instead of $I \xrightarrow{a} A$.
This is really just a switch of perspective: we can recover a from p up to a phase, which is physically unimportant.

Mixed states, categorically

Step 2: switch from



Instead of $A \to A$, may take names $I \to A^* \otimes A$, so no information lost.

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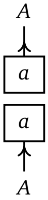
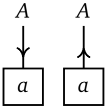


to



Instead of $A \longrightarrow A$, may take names $I \longrightarrow A^* \otimes A$, so no information lost.

A positive matrix is a morphism $I \xrightarrow{m} A^* \otimes A$ that is the name $\ulcorner f^\dagger \circ f \urcorner$ of a positive morphism for some $A \xrightarrow{f} B$. If we can choose $B = I$, we call m a pure state.

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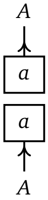

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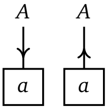

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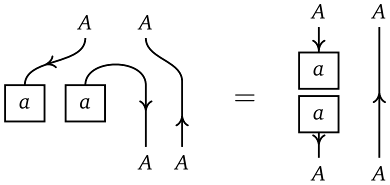
Will sometimes write $\sqrt{m}$ for f to indicate that m has a 'square root' and is hence positive. However, $\sqrt{m}$ is by no means unique.

Mixed states, categorically

Step 3: move from positive matrix $I \xrightarrow{m} A^* \otimes A$ to multiplication $A^* \otimes A \to A^* \otimes A$ on left with m; compare Cayley embedding.



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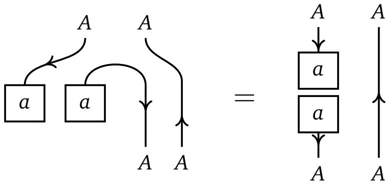
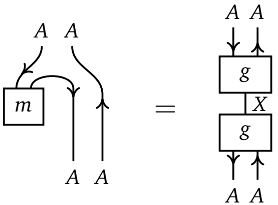


Loses no information:

In **FHilb**, if a morphism $I \xrightarrow{m} A^* \otimes A$ satisfies

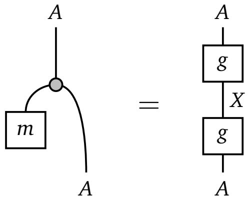


then it is a positive matrix.

Mixed states, categorically

Step 4: Recognize pants, upgrade to arbitrary Frobenius structure.

A mixed state of a dagger Frobenius structure $(A, \ , \ )$ in a monoidal dagger category is a morphism $I \xrightarrow{m} A$ with



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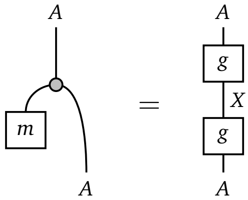

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Will sometimes write $\sqrt[\circ]{m}$ instead of g, even though not unique.

Examples of mixed states

- Recall pair of pants on $A = \mathbb{C}^n$ in **FHilb** is n-by-n matrices. Mixed states are n-by-n matrices m satisfying $m = \sqrt{m}^\dagger \circ \sqrt{m}$ for some n-by-m matrix $\sqrt{m}$: precisely density matrices.

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- Special dagger Frobenius structure in **Rel** correspond to groupoids $\mathbf{G}$. Mixed states are subsets R closed under inverses, and such that $g \in R$ implies $\mathrm{id}_{\mathrm{dom}(g)} \in R$.

What are the morphisms?

Individual morphisms are physical processes; free or controlled time evolution, preparation, or measurement. Should take (mixed) states to (mixed) states, be determined by behaviour on (mixed) states.

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Let $(A, ,)$ and $(B, ,)$ be dagger Frobenius structures in dagger monoidal category. A positive map is morphism $A \xrightarrow{f} B$ such that $I \xrightarrow{f \circ m} B$ is mixed state when $I \xrightarrow{m} A$ is mixed state.

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Warning: different from *positive-semidefinite* morphisms $f = g^\dagger \circ g$, abbreviated to *positive morphisms*.

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Specifically, $f \otimes \mathrm{id}_E$ should be positive map for any Frobenius structure E and any positive map $A \xrightarrow{f} B$. Might only be interested in A, but can never be sure it's isolated from environment E.

Let $(A, \ , \ )$ and $(B, \ , \ )$ be dagger Frobenius structures in a dagger monoidal category. Completely positive map is morphism $A \xrightarrow{f} B$ with $f \otimes \mathrm{id}_E$ is positive map for any dagger Frobenius structure $(E, \ , \ )$.

Examples of completely positive maps

Completely positive maps in **FHilb**:

- Unitary evolution: letting an n-by-n matrix m evolve freely along unitary u to $u^\dagger \circ m \circ u$; can phrase it as $A^* \otimes A \xrightarrow{u_* \otimes u} A^* \otimes A$ for $A = \mathbb{C}^n$.

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Let G and H be the sets of morphisms of groupoids $\mathbf{G}$ and $\mathbf{H}$. A relation $G \to H$ is completely positive if and only if it respects inverses: $g \sim h$ implies $g^{-1} \sim h^{-1}$ and $\mathrm{id}_{\mathrm{dom}(g)} \sim \mathrm{id}_{\mathrm{dom}(h)}$.

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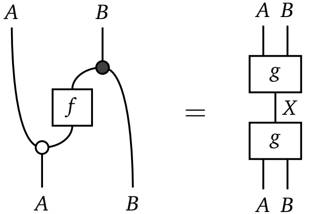
Lemma: In a positively monoidal braided dagger category, if $f : (A, \curlywedge_{\circ}, \circ) \to (B, \curlywedge_{\bullet}, \bullet)$ is completely positive, then



for some object X and some morphism $A \otimes B \xrightarrow{g} X$.

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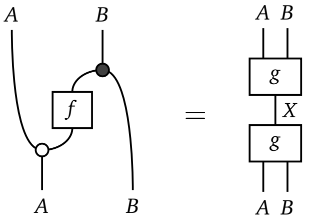

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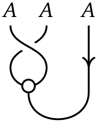
This is called the CP–condition.

Categories of completely positive maps

Proof. Let $E = A \otimes A^*$ be pair of pants, define $I \xrightarrow{m} A \otimes E$ as:



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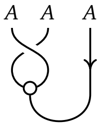
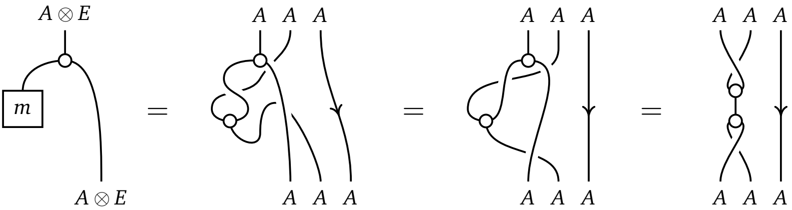


Then m is a mixed state:



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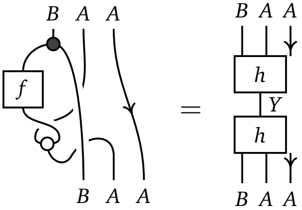
Since f is completely positive, so $(f \otimes \mathrm{id}_E) \circ m$ is a mixed state:



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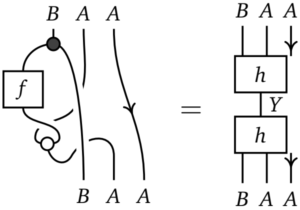

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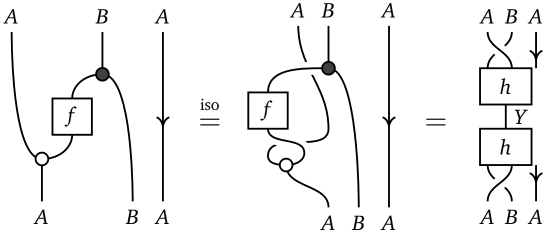

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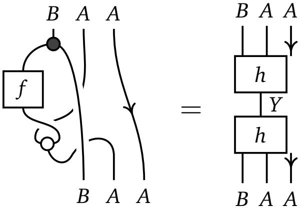

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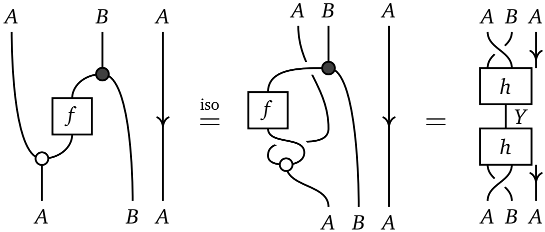
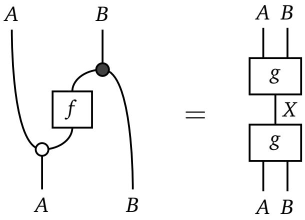


CP–condition then follows from positively monoidal. □

The CP condition



Striking similarity to oracles, Frobenius law.

Object X is also called the ancilla system.
Map g is called a Kraus morphism, written $\sqrt[\circ]{f}$ although not unique.

Will now prove converse; need to show CP–condition well-behaved.

CP maps compose

Lemma: In a monoidal dagger category, let $(A, \ , \ )$, $(B, \ , \ )$, and $(C, \ , \ )$ be special dagger Frobenius structures. If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ satisfy the CP condition, so does $g \circ f$.

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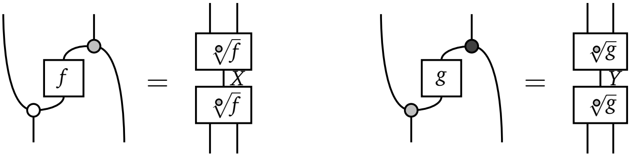

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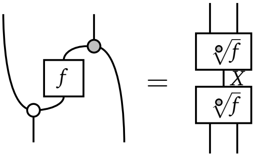
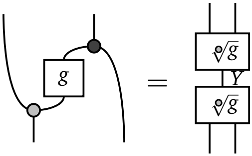
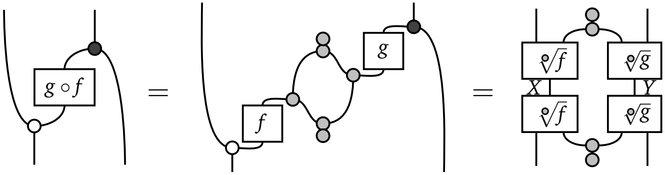




Then we perform the following calculation:



Tensor products of CP maps

Lemma: If $(A, \;, \;) \xrightarrow{f} (B, \;, \;)$ and $(C, \;, \;) \xrightarrow{g} (D, \;, \;)$ are maps between dagger Frobenius structures in a braided monoidal dagger category that satisfy CP–condition, then so is
$(A, \;, \;) \otimes (C, \;, \;) \xrightarrow{f \otimes g} (B, \;, \;) \otimes (D, \;, \;)$.

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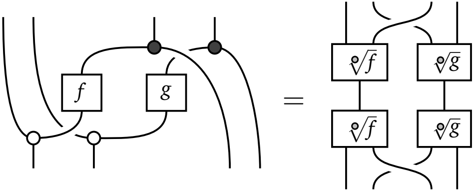
Proof. Suppose $\sqrt[\circ]{f}$ and $\sqrt[\circ]{g}$ are Kraus morphisms for f and g. Then:



Stinespring's theorem

Theorem: Let $(A, \ , \ )$ and $(B, \ , \ )$ be special dagger Frobenius structures, $A \xrightarrow{f} B$ morphism in braided monoidal dagger category that is positively monoidal. The following are equivalent:

(a) f is completely positive;

(b) $f \otimes \mathrm{id}_E$ is positive map for all $E = (X^* \otimes X, \ , \ )$;

(c) f satisfies the CP–condition.

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Proof. (a) $\Rightarrow$ (b) clear; (b) $\Rightarrow$ (c) already shown; (c) $\Rightarrow$ (a) follows from previous two lemmas. $\square$

The CP construction

Turn compact dagger category **C** modeling pure states into new compact dagger category CP[**C**] of mixed states.

Let **C** be a monoidal dagger category. Define a new category CP[**C**] as follows: objects are special dagger Frobenius structures in **C**, and morphisms are completely positive maps.

CP preserves tensors

If **C** is a braided monoidal dagger category, then CP[**C**] is a monoidal category:

- the tensor product of objects is product comonoid;
- the tensor product of morphisms is well-defined by lemma;
- the tensor unit is I with multiplication $I \otimes I \xrightarrow{\rho_I} I$ and unit $I \xrightarrow{\mathrm{id}_I} I$;
- the coherence isomorphisms α, λ, and ρ are inherited from **C**.

If **C** is a symmetric monoidal category, then so is CP[**C**].

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If $\mathbf{C}$ is a braided monoidal dagger category, then $\mathrm{CP}[\mathbf{C}]$ is a monoidal category:

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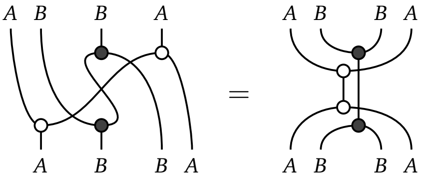
If $\mathbf{C}$ is a symmetric monoidal category, then so is $\mathrm{CP}[\mathbf{C}]$.

Proof. If $\mathbf{C}$ symmetric, swap maps are CP by Frobenius:



Hence, in that case, $\mathrm{CP}[\mathbf{C}]$ is symmetric monoidal. □

CP preserves daggers

Let $(A, \;, \;)$ and $(B, \;, \;)$ be special dagger Frobenius structures in a braided monoidal dagger category. If $A \xrightarrow{f} B$ satisfies CP–condition, so does $B \xrightarrow{f^\dagger} A$.

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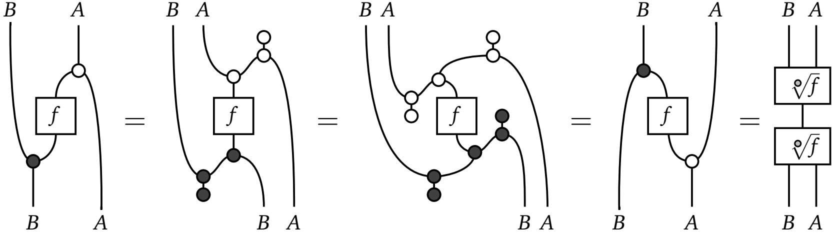
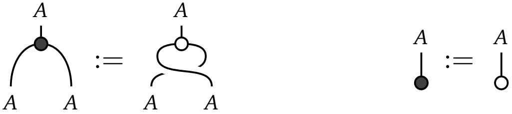
Proof.



CP preserves duals

Let $(A, \ , \ )$ be a special dagger Frobenius structure in a braided monoidal dagger category $\mathbf{C}$, and:



Then $(A, \ , \ ) \dashv (A, \ , \ )$ in $\mathrm{CP}[\mathbf{C}]$.

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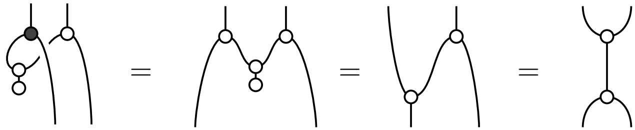
Proof. Define $\;:=\;: I \to R \otimes L$.



Also $\;:=\;: L \otimes R \to I$ is CP.
Because composition in $\mathrm{CP}[\mathbf{C}]$ is as in $\mathbf{C}$, snake equations come down precisely to the Frobenius law. Thus $L \dashv R$ in $\mathrm{CP}[\mathbf{C}]$. □

CP summary

$\mathbf{C}$	$\mathrm{CP}[\mathbf{C}]$
monoidal dagger category	category
braided monoidal dagger category	monoidal category right duals
symmetric monoidal dagger category	compact category
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Examples:

- CP[**FHilb**]: fin-dim C*-algebras and completely positive maps
- CP[**Rel**]: groupoids and inverse-respecting relations

Classical structures

If $\mathbf{C}$ is braided monoidal dagger, then category $\mathrm{CP}_c[\mathbf{C}]$ has:

- as objects classical structures in $\mathbf{C}$
- as morphisms completely positive maps.

If $\mathbf{C}$ is compact, so is $\mathrm{CP}_c[\mathbf{C}]$; any object in $\mathrm{CP}_c[\mathbf{C}]$ is self-dual.

If $\mathbf{C}$ models pure state quantum mechanics, and $\mathrm{CP}[\mathbf{C}]$ mixed state quantum mechanics, then $\mathrm{CP}_c[\mathbf{C}]$ models statistical mechanics.

Stochastic matrices

$\mathrm{CP_c}[\mathbf{FHilb}]$ is monoidally equivalent to:

- objects are natural numbers
- morphisms are m-by-n matrices of nonnegative real entries

Maps that preserve counit are matrices whose rows sum to one: stochastic matrices.

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Consistent with comomonoid homomorphisms of classical structures:

- every column has single entry 1 and 0s elsewhere
- deterministic maps within stochastic setting

Broadcasting

Compact dagger categories have no uniform copying/deleting.
However, doesn't yet mean they model quantum mechanics.

- classical mechanics might have copying
- quantum mechanics might not have copying
- but statistical mechanics has no copying either

Rather: impossibility of broadcasting unknown mixed states.

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Rather: impossibility of broadcasting unknown mixed states.

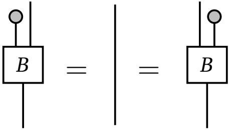
First make sure that there exist 'discarding' maps $A \to I$ in CP[$\mathbf{C}$]:

Lemma: Let $(A, \ , \ )$ be dagger Frobenius structure in braided monoidal dagger category **C**. Then is completely positive. If $(A, \ , \ )$ is classical structure, then is completely positive.

Proof. Verifying CP condition for is easy. CP condition for commutative rewrites into positive form using spider theorem.

No broadcasting

Let **C** be braided monoidal dagger category. A broadcasting map for object $(A, \curlywedge, \downarrow)$ of CP[**C**] is morphism $A \xrightarrow{B} A \otimes A$ in CP[**C**] satisfying:



Object $(A, \curlywedge, \downarrow)$ is broadcastable if it allows a broadcasting map.

Note: concerns just single object, so weaker than uniform copying.

No broadcasting in **FHilb**

Let **C** be a braided monoidal dagger category. Classical structures are broadcastable objects in CP[**C**].

Proof.  satisfies CP condition. □

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- In **FHilb** converse holds: no-broadcasting theorem.
  So dagger Frobenius structure broadcastable iff classical structure.

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  So dagger Frobenius structure broadcastable iff classical structure.
- Not so in **Rel**! Call category totally disconnected when only morphisms are endomorphisms.

Broadcasting in **Rel**

Broadcastable objects in CP[**Rel**] are totally disconnected groupoids.

Proof. If **G** totally disconnected, then $G \xrightarrow{B} G \times G$ given by

$$B = \{\big(g, (\mathrm{id}_{\mathrm{dom}(g)}, g)\big) \mid g \in G)\} \cup \{\big(g, (g, \mathrm{id}_{\mathrm{dom}(g)})\big) \mid g \in G\}$$

is broadcasting map.

Converse: use that broadcasting means

$$\begin{aligned} \{(g,g) \mid g \in G\} &= \{(g,h) \mid (g, (\mathrm{id}_{\mathrm{cod}(h)}, h)) \in B\} \\ &= \{(g,h) \mid (g, (h, \mathrm{id}_{\mathrm{dom}(h)})) \in B\}. \end{aligned}$$

Summary

- Completely positive maps:
  pure states/evolutions vs mixed ones
- Categories of completely positive maps:
  everything happily in one category
- Classical structures:
  operational view, broadcasting