Categories and Quantum Informatics: Complete posivity

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Up to now, we have only considered categorical models of *pure* states. But if we really want to take grouping systems together seriously as a primitive notion, we should also care about *mixed* states. This means we have to add another layer of structure to our categories. This chapter studies a beautiful construction with which we don't have to step outside the realm of compact dagger categories after all, and brings together all the material from previous chapters. It revolves around *completely positive maps*.

Section 7.1 first abstracts this notion from standard quantum theory to the categorical setting. In Section 7.2, we then reformulate such morphisms into a convenient condition, and present the central *CP construction*. In the resulting categories, classical and quantum systems live on equal footing. We also prove an abstract version of Stinespring's theorem, characterizing completely positive maps in operational terms. Subsequently we consider the subcategory containing only classical systems. Section 7.3 considers no-broadcasting theorems as mixed versions of the no-cloning theorem of Section 4.2.

7.1 Completely positive maps

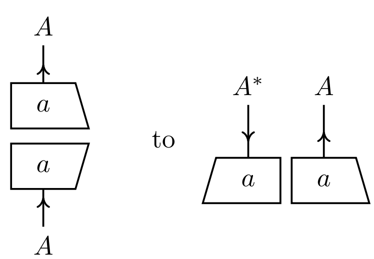
In this section we investigate evolution of mixed states of systems, by which we mean procedures that send mixed states to mixed states. First, we define mixed states themselves, and then extrapolate. It turns out that the evolutions we are after correspond to completely positive maps, and mixed states are simply completely positive maps from the tensor unit I to a system.

Mixed states

So far we have defined a *pure state* as a morphism $I \xrightarrow{a} A$. To eventually arrive at a definition of mixed state that makes sense in arbitrary compact dagger categories, we proceed in four steps.

The first step is to consider the induced morphism $p = a \circ a^\dagger \colon A \longrightarrow A$ instead of $I \xrightarrow{a} A$. This is really just a switch of perspective, as we can recover a from p up to a physically unimportant phase.

The second step is to switch from



(7.1)

Instead of a morphism $A \longrightarrow A$ in a compact dagger category, we may equivalently work with matrices $I \longrightarrow A^* \otimes A$ by taking names (see Definition 3.3). That is, a *matrix* is a state on $A^* \otimes A$. So no information is lost in this step; morphisms of the form $A \xrightarrow{a \circ a^\dagger} A$ turn out to correspond to certain so-called *positive matrices* $I \xrightarrow{m} A^* \otimes A$.

Definition 7.1 (Positive matrix, pure state)**.** A *positive matrix* is a morphism $I \xrightarrow{m} A^* \otimes A$ that is the name $\ulcorner f^\dagger \circ f \urcorner$ of a positive morphism for some $A \xrightarrow{f} B$. If we can choose $B = I$, we call m a *pure state*.

We will sometimes write $\sqrt{m}$ for f to indicate that m has a 'square root' and is hence positive. However, notice that such a morphism $\sqrt{m}$ is by no means unique.

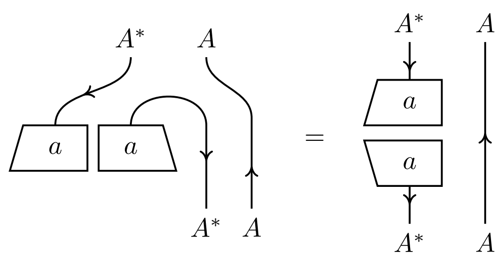
Example 7.2. In our example categories:

- Positive matrices in **FHilb** come down to linear maps $\mathbb{C} \longrightarrow \mathbb{M}_n$ that send 1 to a positive matrix $f \in \mathbb{M}_n$; use Example 4.12. Pure states correspond to positive matrices of rank at most 1, that is, those of the form $|a\rangle\langle a|$ for a vector $a \in \mathbb{C}^n$. This is precisely what we called a *pure state* in Hilbert space quantum mechanics.

- Positive matrices $I \longrightarrow A \times A$ in **Rel** come down to subsets $R \subseteq A \times A$ that are symmetric and satisfy aRa when aRb. The pure states are of the form $R = X \times X \subseteq A \times A$ for subsets $X \subseteq A$.

So far we have merely reformulated pure states. We now generalise from pure states to mixed states. The final two steps of our process reformulate and generalize this further.

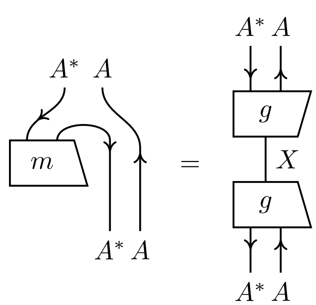
The third step is a conceptual leap, that moves from the positive matrix $I \xrightarrow{m} A^* \otimes A$ to the map $A^* \otimes A \longrightarrow A^* \otimes A$ that multiplies on the left with the matrix m; compare also the Cayley embedding of Proposition 4.13:



(7.2)

This morphism is clearly positive. The following lemma shows the converse, so that this reformulation again loses no information.

Lemma 7.3. *If a morphism $I \xrightarrow{m} A^* \otimes A$ in* **FHilb** *satisfies*

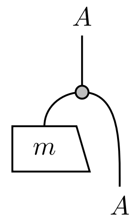


(7.3)

then it is a positive matrix.

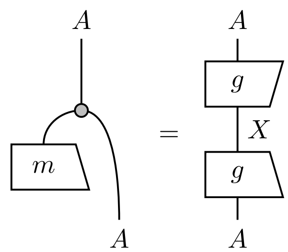
Proof. For any morphism $H \xrightarrow{f} H$ in **FHilb**, it follows from the Kronecker product (30) that $f \otimes \mathrm{id}_K$ is a block diagonal matrix; the $\dim(K)$ many diagonal blocks are simply the matrix of f. Hence $f \otimes \mathrm{id}_K$ is diagonalizable precisely when f is (and $\dim(K) > 0$), and the eigenvalues of $f \otimes \mathrm{id}_K$ are simply ($\dim(K)$ many copies of) the eigenvalues of f. In particular, if $\dim(K) > 0$ then $f \otimes \mathrm{id}_K$ is positive precisely when f is. Thus if (7.3) holds, then $m = \ulcorner f \urcorner$ for some positive morphism f, making m a positive matrix. $\square$

In the fourth and final step, we recognize in the left-hand sides of (7.2) and (7.3) the multiplication of the pair of pants monoid (see Lemma 5.9). Upgrade the pair of pants to an arbitrary Frobenius structure multiplication to obtain the generalization:



We have arrived at our definition of a mixed state.

Definition 7.4 (Mixed state)**.** A *mixed state* of a dagger Frobenius structure (A, μ, η) in a monoidal dagger category is a morphism $I \xrightarrow{m} A$ satisfying



(7.4)

for some object X and some morphism $A \xrightarrow{g} X$.

We will sometimes write $\sqrt[\circ]{m}$ instead of g, even though it is not unique.

Example 7.5. In our example categories:

- Recall from Example 4.12 that the pair of pants monoid on $A = \mathbb{C}^n$ in **FHilb** is precisely the algebra of n-by-n matrices. The mixed states come down to n-by-n matrices m satisfying $m = \sqrt{m}^\dagger \circ \sqrt{m}$ for some n-by-m matrix $\sqrt{m}$. Those are precisely the *mixed states*, or *density matrices*.

  In general, recall from Theorem 5.29 dagger Frobenius structures in **FHilb** correspond to finite-dimensional operator algebras A. The mixed states $I \longrightarrow A$ come down to those elements $a \in A$ satisfying $a = b^* b$ for some $b \in A$; these are usually called the *positive* elements.

- Recall from Theorem 5.37 that special dagger Frobenius structure in **Rel** correspond to groupoids $\mathbf{G}$. Mixed states come down to subsets R of the morphisms of $\mathbf{G}$ such that the relation defined by $g \sim h$ if and only if $h = r \circ g$ for some $r \in R$ is positive. This boils down to: R is closed under inverses, and if $g \in R$, then also $\mathrm{id}_{\mathrm{dom}(g)} \in R$.

Completely positive maps

As we have seen in Sections 5.3 and 5.4, we may think of Frobenius structures as comprising *observables*, *i.e.* self-adjoint operators $A \longrightarrow A$. In this section we will develop the accompanying notion of morphism. Individual morphisms are regarded as physical processes, such as free or controlled time evolution, preparation, or measurement. They should therefore take (mixed) states to (mixed) states, and be completely determined by their behaviour on (mixed) states. Such morphisms are abbreviated to *positive maps*, because they preserve positive elements; just as a linear map is one that preserves linear combinations.

Definition 7.6 (Positive map)**.** Let (A, μ, η) and (B, μ, η) be dagger Frobenius structures in a dagger monoidal category. A *positive map* is a morphism $A \xrightarrow{f} B$ such that $I \xrightarrow{f \circ m} B$ is a mixed state whenever $I \xrightarrow{m} A$ is a mixed state.

Warning: note the difference with *positive-semidefinite* morphisms $f = g^\dagger \circ g$, that we have abbreviated to *positive morphisms* in Definition 2.11; luckily contexts will hardly arise where it's difficult to differentiate between the two notions.

Instead of mixed states $I \xrightarrow{m} A$ and morphisms $A \xrightarrow{f} B$, we could dualize to effects $A \longrightarrow I$ and morphisms $B \xrightarrow{f^\dagger} A$. Rather than f mapping states to states, $f^\dagger$ will map effects to effects in the other direction. This is the difference between the *Schrödinger picture* and the *Heisenberg picture*. In the former, observables stay fixed, while states evolve over time. In the latter, states stay fixed, while observables (effects) evolve over time. Although both pictures are equivalent, we will mostly adhere to the Schrödinger one.

However, positive maps are not yet the 'right' morphisms, precisely because they forget about the main premise of this book: always take compound systems into account! If f and g are physical channels, then we would like $f \otimes g$ to be a physical channel, too. Specifically, we would like $f \otimes \mathrm{id}_E$ to be a positive map for any Frobenius structure E and any positive map $A \xrightarrow{f} B$. We might only be interested in the system A, but we can never be completely sure that we have isolated it from the environment E. To account for the dynamics of such *open systems* we have to use so-called *completely positive* maps.

Definition 7.7 (Completely positive map)**.** Let $(A, ,)$ and $(B, ,)$ be dagger Frobenius structures in a dagger monoidal category. A *completely positive map* is a morphism $A \xrightarrow{f} B$ such that $f \otimes \mathrm{id}_E$ is a positive map for any dagger Frobenius structure $(E, ,)$.

The next two subsections investigate the completely positive maps in our example categories **FHilb** and **Rel**.

Evolution and measurement

In the category **FHilb**, Definition 7.7 is precisely the traditional definition of completely positive maps; that's how we engineered it. They bring evolution, measurement, and preparation on an equal footing.

Example 7.8. The following are completely positive maps in **FHilb**:

- *Unitary evolution*: letting an n-by-n matrix m evolve freely along a unitary u to $u^\dagger \circ m \circ u$ is a completely positive map. With Example 4.12 we can phrase it as the map $A^* \otimes A \xrightarrow{u_* \otimes u} A^* \otimes A$, from a pair of pants Frobenius structure to itself, where $A = \mathbb{C}^n$.

- Let $A \xrightarrow{p_1, \ldots, p_n} A$ form a projection-valued measure with n outcomes. Then the function $\mathbb{C}^n \longrightarrow A^* \otimes A$ that sends the computational basis vector $|i\rangle$ to p_i is a completely positive map, from the classical structure $\mathbb{C}^n$ to the pair of pants Frobenius structure $A^* \otimes A$.

  Note the direction: that of the Heisenberg picture. In Lemma 7.21 below, we will see that the choice of direction is arbitrary.

- More generally, if $A \xrightarrow{p_1, \ldots, p_n} A$ is a positive operator-valued measure, $|i\rangle \mapsto p_i$ is still a completely positive map $\mathbb{C}^n \longrightarrow A^* \otimes A$.

  In fact, the converse holds, too: if $\mathbb{C}^n \xrightarrow{p} A^* \otimes A$ is a completely positive map that preserves units, then $\{p(|1\rangle), \ldots, p(|n\rangle)\}$ is a positive operator-valued measure. Hence a completely positive map from a classical structure to a pair of pants Frobenius structure corresponds to a *measurement*.

- A completely positive map $\mathbb{C} \longrightarrow A^* \otimes A$ is precisely (the *preparation* of) a mixed state. This example generalizes to arbitrary braided monoidal dagger categories.

- More generally, suppose we would like to prepare one of n mixed states $A \xrightarrow{m_i} A$, depending on some input parameter $i = 1, \ldots, n$. We can phrase this as the map $\mathbb{C}^n \longrightarrow A^* \otimes A$ given by $|i\rangle \mapsto m_i$, which is completely positive. We can therefore regard a completely positive map from a classical structure to a pair of pants Frobenius structure, as a *controlled preparation*.

Inverse-respecting relations

In our other running example, the category **Rel** of sets and relations, special dagger Frobenius structures correspond to groupoids by Theorem 5.37. Just like completely positive maps in **FHilb** only care about positivity, and not the multiplication of the involved Frobenius structure, completely positive maps in **Rel** only care about inverses, and not the multiplication of the groupoids.

Definition 7.9 (Inverse-respecting relation)**.** Let G and H be the sets of morphisms of groupoids **G** and **H**. A relation $G \xrightarrow{R} H$ is said to *respect inverses* when gRh implies $g^{-1}Rh^{-1}$ and $\mathrm{id}_{\mathrm{dom}(g)}R\mathrm{id}_{\mathrm{dom}(h)}$.

Proposition 7.10. *A morphism* $\mathbf{G} \xrightarrow{R} \mathbf{H}$ *in the category* **Rel** *is completely positive if and only if it respects inverses.*

Proof. First assume R respects inverses. Let **K** be any groupoid; write G, H, K for the sets of morphisms of $\mathbf{G}, \mathbf{H}, \mathbf{K}$. Suppose $S \subseteq G \times K$ that is a mixed state, that is, by Example 7.5, that S is closed under inverses and identities. Then $(R \times \mathrm{id}) \circ S$ is $\{(h,k) \in H \times K \mid \exists g \in G\colon (g,k) \in S, (g,k) \in R\}$. This is clearly closed under inverses and identities again, so R is completely positive.

Conversely, suppose R is completely positive. Take $\mathbf{K} = \mathbf{G}$, and let $a \xrightarrow{g} b$ be a morphism in **G**. Define $S = \{(g,g), (g^{-1}, g^{-1}), (\mathrm{id}_a, \mathrm{id}_a), (\mathrm{id}_b, \mathrm{id}_b)\}$. This is a mixed state, hence so is $(R \times \mathrm{id}) \circ S$, which equals

$$\{(h,g) \mid gRh\} \cup \{(h, g^{-1}) \mid g^{-1}Rh\} \cup \{(h, \mathrm{id}_a) \mid \mathrm{id}_a Rh\} \cup \{(h, \mathrm{id}_b) \mid \mathrm{id}_b Rh\}.$$

If gRh, it follows that $g^{-1}Rh^{-1}$, and $\mathrm{id}_a R \mathrm{id}_{\mathrm{dom}(h)}$, so R respects inverses. □

The characterisation of completely positive maps in **Rel** of the previous proposition is the source of many ways in which **Rel** differs from **FHilb**. In other words, even though we have sketched **Rel** as a model of 'possibilistic quantum mechanics', it is a *nonstandard model* of quantum mechanics. It provides counterexamples to many features that are sometimes thought to be quantum but turn out to be 'accidentally' true in **FHilb**. See for example Section 7.3 later. For another example: a positive map between Frobenius structures in **FHilb**, at least one of which is commutative, is automatically completely positive. The same is not true in **Rel**.

Example 7.11 (The need for complete positivity)**.** The following relation $(\mathbb{Z}, +, 0) \xrightarrow{R} (\mathbb{Z}, +, 0)$ is positive but not completely positive:

$$R = \{(n,n) \mid n \geq 0\} \cup \{(n, -n) \mid n \geq 0\} = \{(|n|, n) \mid n \in \mathbb{Z}\}.$$

Hence complete positivity is strictly stronger than (mere) positivity.

Proof. Let $I \xrightarrow{m} \mathbb{Z}$ be a nonzero mixed state. We may equivalently consider the subset $S = \{n \in \mathbb{Z} \mid (*, n) \in m\} \subseteq \mathbb{Z}$ satisfying $0 \in S$ and $S^{-1} \subseteq S$ by Proposition 7.10. Now $(*, n) \in R \circ m$ if and only if $|n| \in S$, if and only if $-n, n \in S$, if and only if $(*, -n) \in R \circ m$. Trivially also $(*, 0) \in R \circ m$. Thus $R \circ m$ is a mixed state, and R is a positive map.

However, R is not completely positive because it clearly does not respect inverses: $(1,1) \in R$ but not $(-1,-1) \in R$. □

7.2 Categories of completely positive maps

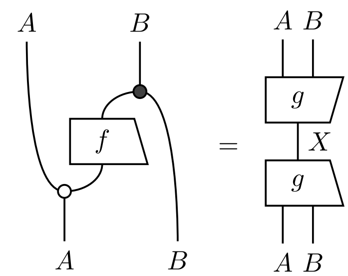
This section describes the main construction of the chapter: starting with a category of pure states, it constructs the corresponding category of mixed states. We start by characterizing Definition 7.7 of completely positive maps from an operational form into a more convenient structural form.

The CP condition

A mixed state of a Frobenius structure $(A, ,)$ is the special case of a completely positive map $I \longrightarrow A$, as we saw illustrated in Example 7.5. The condition characterizing when a map is completely positive that we will end up with is a generalization of equation (7.4).

For this proof, we will need the same mild assumption as we did in the third step of Section 7.1. Namely that if $(A, ,)$ is a dagger Frobenius structure, B is not a zero object, and $f \otimes \mathrm{id}_B$ for $A \otimes B \xrightarrow{f} A \otimes B$ is a positive morphism (*i.e.* is of the form $g^\dagger \circ g$ for some g), then f itself is already positive. Let's call a category with this property *positively monoidal*. This requirement is satisfied when is an invertible scalar, for example; it is also satisfied when A is a zero object. Intuitively, this requirement demands that the dimension of a Frobenius structure is zero or invertible, which is the case in both of our running example categories **FHilb** and **Rel**.

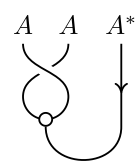
Lemma 7.12 (CP condition). *Let $(A, ,)$ and $(B, ,)$ be dagger Frobenius structures in a braided monoidal dagger category that is positively monoidal. If $A \xrightarrow{f} B$ is completely positive, then*



(7.5)

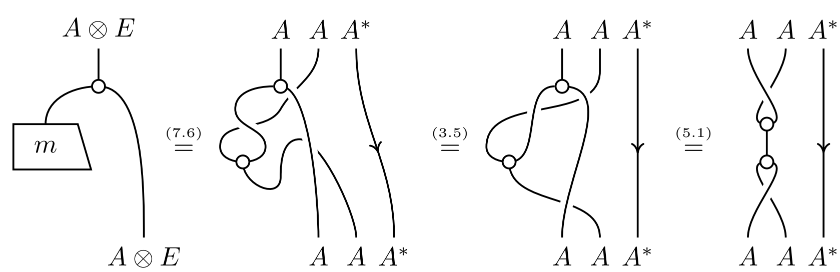
for some object X and some morphism $A \otimes B \xrightarrow{g} X$.

Proof. Notice that A supports a dagger Frobenius structure and hence has a dagger dual object A^* by Theorem 5.15. Let E be the pair of pants monoid $A \otimes A^*$, and define $I \xrightarrow{m} A \otimes E$ as:

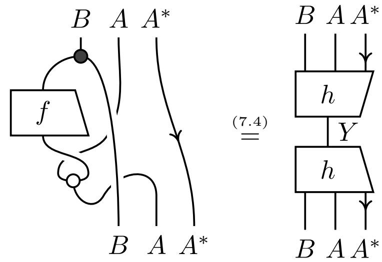


(7.6)

Then m is a mixed state:

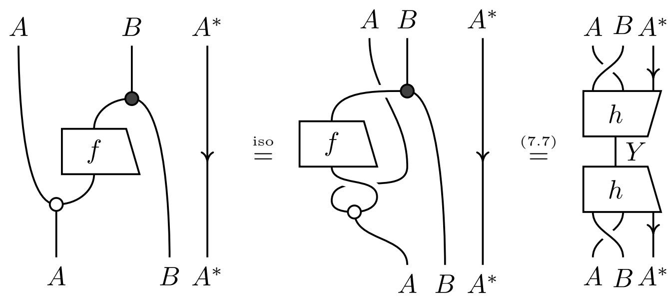


The first equality just unfolds the definition of m and the composite Frobenius structure on $A \otimes E$, the second equality uses a snake equation 3.5, whereas the third equality uses the Frobenius law. Now $(f \otimes \mathrm{id}_E) \circ m$ is a mixed state:



(7.7)

for some object Y and morphism h. Hence:

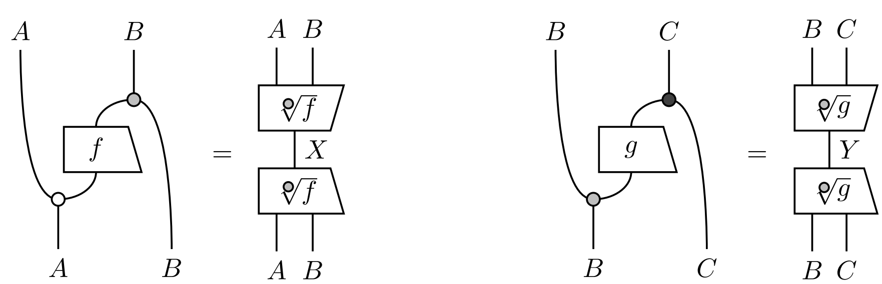


Because the category is positively monoidal, equation (7.5) now follows. $\square$

Equation (7.5) is called the *CP condition.* Notice its similarity to the Frobenius identity (5.4), and also to the oracles of Definition 6.12; the latter required the left-hand side to be unitary, whereas the CP condition requires it to be positive. The object X is also called the *ancilla system*. The map g is called a *Kraus morphism*, and is also written $\sqrt[\circ]{f}$, although it is not unique. The mixed state $(f \otimes \mathrm{id}) \circ m$ is also called the *Choi-matrix*; it is the transform under the Choi-Jamiołkowski isomorphism of the completely positive map. We will shortly prove the converse of the previous lemma, but first need two preparatory lemmas showing that the CP condition is well-behaved with respect to composition and tensor products.

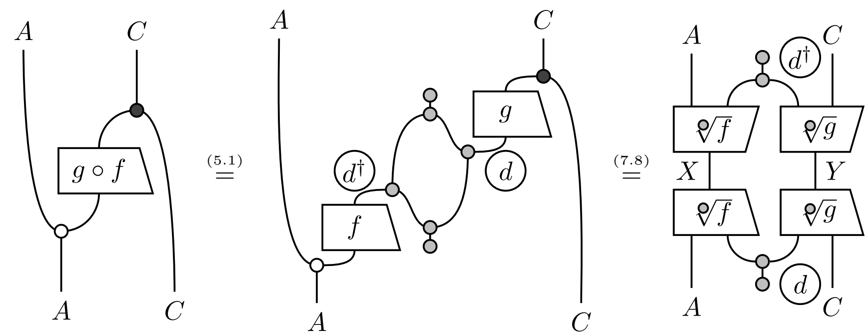
Lemma 7.13 (CP maps compose)**.** *Let $(A, ,)$, $(B, ,)$, and $(C, ,)$ be dagger Frobenius structures in a monoidal dagger category. Assume $d^\dagger \bullet \bullet d = \mathrm{id}_B$ for some scalar d. If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ satisfy the CP condition (7.5), then so does $A \xrightarrow{g \circ f} C$.*

Proof. Say:



(7.8)

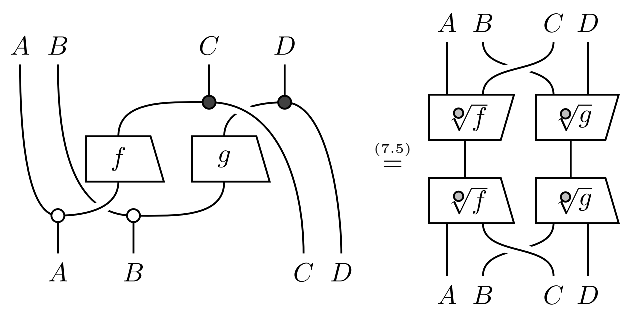
for objects X, Y and morphisms $\sqrt{f}$, $\sqrt{g}$. Then:



This uses the Frobenius law to insert a 'handle' $d^\dagger \bullet \bullet d$. $\square$

Lemma 7.14 (Product CP maps)**.** *If* $(A, \cdot, \cdot) \xrightarrow{f} (B, \cdot, \cdot)$ *and* $(C, \cdot, \cdot) \xrightarrow{g} (D, \cdot, \cdot)$ *are maps between dagger Frobenius structures in a braided monoidal dagger category that satisfy the CP condition* (7.5)*, then so is* $(A, \cdot, \cdot) \otimes (C, \cdot, \cdot) \xrightarrow{f \otimes g} (B, \cdot, \cdot) \otimes (D, \cdot, \cdot)$.

Proof. Suppose $\sqrt{f}$ and $\sqrt{g}$ are Kraus morphisms for f and g. Then:



This proves the lemma. $\square$

Stinespring's theorem

We can now prove that the CP condition characterizes completely positive maps. Notice that the proof of Lemma 7.12 did not need arbitrary ancilla systems E, and pair of pants monoids sufficed. The following theorem will also record that.

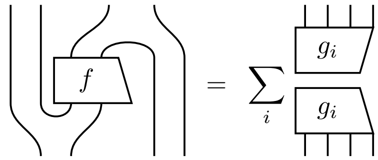
Theorem 7.15 (Stinespring)**.** *Let* $(A, \cdot, \cdot)$ *and* $(B, \cdot, \cdot)$ *be special dagger Frobenius structures and* $A \xrightarrow{f} B$ *a morphism in a braided monoidal dagger category that is positively monoidal. The following are equivalent:*

- *(a)* f *is completely positive;*
- *(b)* $f \otimes \mathrm{id}_E$ *is a positive map for all objects* X*, where* $E = (X^* \otimes X, \cdot, \cup)$*;*
- *(c)* f *satisfies the CP condition* (7.5).

Proof. Clearly (a) implies (b). Lemma 7.12 shows that (b) implies (c). Finally, to show that (c) implies (a), let $I \xrightarrow{m} (A, \cdot, \cdot) \otimes (E, \cdot, \cdot)$ be a mixed state. Then m is a completely positive map and so satisfies the CP condition. Hence, by Lemmas 7.13 and 7.14, also $(f \otimes \mathrm{id}_E) \circ m$ satisfies the CP condition and is thus a mixed state. $\square$

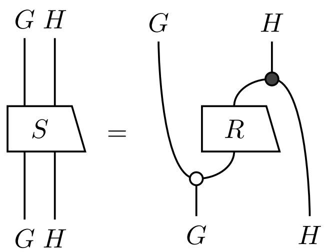
Example 7.16. Let's unpack what the previous theorem says in our example categories **FHilb** and **Rel**.

- For a completely positive map $A^* \otimes A \xrightarrow{f} A^* \otimes A$ in **FHilb**, for $A = \mathbb{C}^n$, so on n-by-n matrices, the CP condition (7.5) becomes

  by choosing a basis $|i\rangle$ for the ancilla system and indexing the Kraus morphisms g_i accordingly. Putting a cap on the top left and a cup on the bottom right we see that this is equivalent to $f(m) = \sum_i f_i^\dagger \circ m \circ f_i$ for matrices m. This generalizes Example 7.8, and we recognize the previous theorem as Stinespring's theorem, or rather, Choi's finite-dimensional version of it.

- In **Rel**, a relation $\mathbf{G} \xrightarrow{R} \mathbf{H}$ between groupoids satisfies the CP condition when the relation

  $= \{((g_1, h_1), (g_2, h_2)) \mid (g_2^{-1} \circ g_1) R (h_2 \circ h_1^{-1})\}$

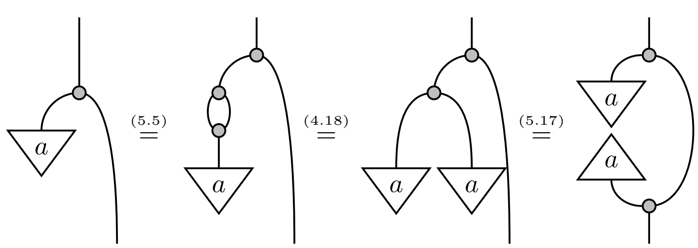
  is positive. This is the case when it is symmetric and satisfies $(g, h)S(g, h)$ when $(g, h)S(g', h')$, matching Proposition 7.10 as follows.

  First, S is symmetric when $(g_2^{-1} \circ g_1)R(h_2 \circ h_1^{-1}) \Leftrightarrow (g_1^{-1} \circ g_2)R(h_1 \circ h_2^{-1})$. Taking g_2 and h_1 to be identities shows that this means $gRh \Leftrightarrow g^{-1}Rh^{-1}$ for all $g \in G$ and $h \in H$. Similarly, S satisfies the other property when $(g_2^{-1} \circ g_1)R(h_2 \circ h_1^{-1})$ implies $\mathrm{id}_{\mathrm{dom}(g_1)} R \mathrm{id}_{\mathrm{dom}(h_1^{-1})}$. But this means precisely that gRh implies $\mathrm{id}_{\mathrm{dom}\, g} R \mathrm{id}_{\mathrm{dom}\, h}$.

For another example, we can now prove that copyable states are always completely positive maps, generalizing Example 7.8.

Corollary 7.17. *Any self-conjugate copyable state $I \xrightarrow{a} A$ of a classical structure $(A, ,)$ in a braided monoidal dagger category is a completely positive map.*

Proof. Graphical manipulation:



This used specialness, copyability, self-conjugateness and the Spider Theorem 5.22. □

The CP construction

We are now ready to define the main construction of this chapter. It takes a compact dagger category **C** modeling pure states, and lifts it to a new compact dagger category CP[**C**] of mixed states.

Definition 7.18 (CP construction)**.** Let **C** be a monoidal dagger category. Define a new category CP[**C**] as follows: objects are special dagger Frobenius structures in **C**, and morphisms are morphisms in **C** that satisfy the CP condition (7.5).

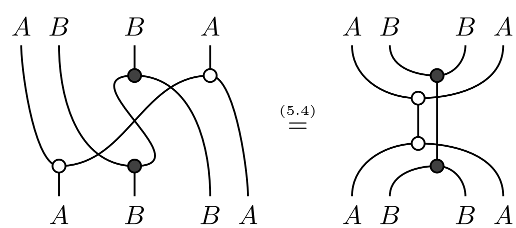
Note that CP[**C**] is indeed a well-defined category: identities in **C** satisfy the CP condition precisely because of the Frobenius law, and Lemma 7.13 shows that composition preserves the CP condition. The CP construction preserves much more than being a category, as we investigate next.

Proposition 7.19 (CP preserves tensors)**.** *If* **C** *is a braided monoidal dagger category, then* CP[**C**] *is a monoidal category:*

- *the tensor product of objects is that of Lemma 4.8;*
- *the tensor product of morphisms is well-defined by Lemma 7.14;*
- *the tensor unit is I with multiplication $I \otimes I \xrightarrow{\rho_I} I$ and unit $I \xrightarrow{\mathrm{id}_I} I$;*
- *the coherence isomorphisms α, λ, and ρ, are inherited from* **C**.

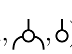
If **C** *is a symmetric monoidal category, then so is* CP[**C**].

Proof. The tensor unit I is a well-defined special dagger Frobenius structure by the coherence theorem. Using these definitions of $\otimes$ and I, the unitary coherence isomorphisms α, λ, and ρ, from **C** trivially satisfy the CP condition. Thus CP[**C**] is a well-defined monoidal category. If **C** is additionally symmetric, the swap maps satisfy the CP condition by the Frobenius law:

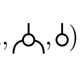
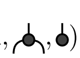


Hence, in that case, CP[**C**] is symmetric monoidal. $\square$

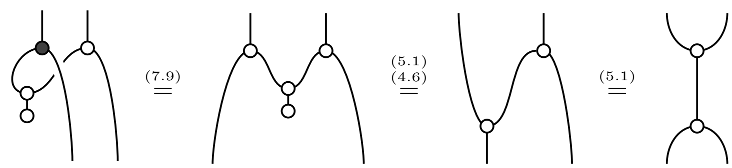
It might look like the following lemma shows that the CP construction fabricates dual objects out of thin air. But note that they were already present in **C** in the sense that Frobenius structures have duals by Theorem 5.15.

Lemma 7.20 (CP constructs duals)**.** *Let* (A, ) *be a special dagger Frobenius structure in a braided monoidal dagger category* **C**. *Define:*

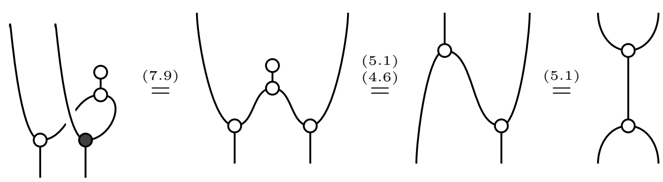
(7.9)

Then (A, ) $\dashv$ (A, ) *in* CP[**C**].

Proof. Easy graphical manipulations show that $(A, \ , \ )$ again satisfies associativity, unitality, the Frobenius law, and specialness. Hence we have two well-defined objects $L := (A, \ , \ )$ and $R = (A, \ , \ )$ of CP[**C**]. Next, define $\ := \ : I \longrightarrow R \otimes L$. We show that this is a well-defined morphism in CP[**C**] by checking the CP condition:



The first equality unfolds definitions and uses naturality of braiding, and the last two apply the Frobenius law and unitality. Similarly, $\ := \ : L \otimes R \longrightarrow I$ is completely positive:

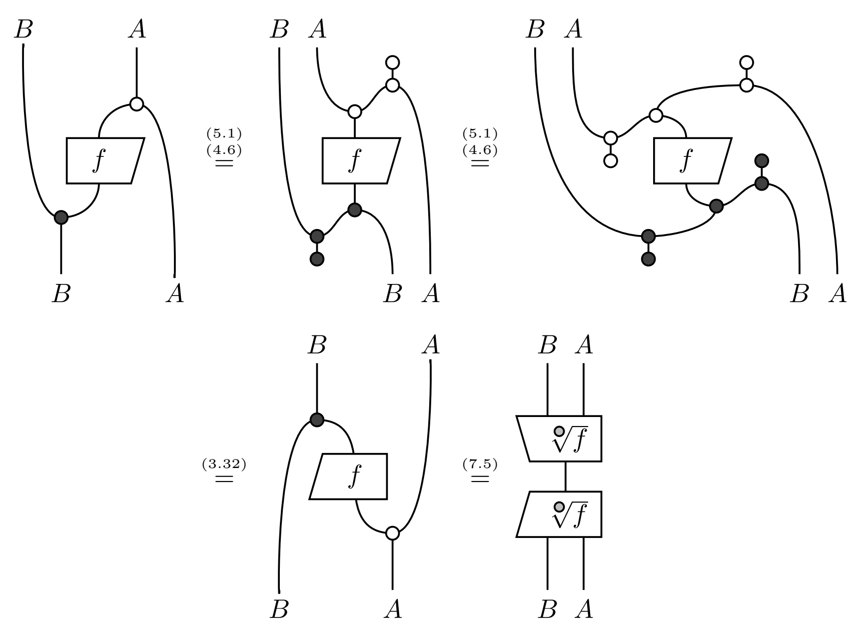


Because composition in CP[**C**] is as in **C**, the snake equations come down precisely to the Frobenius law. Thus and witness $L \dashv R$ in CP[**C**]. $\square$

The next lemma sets up a perfect duality between the Schrödinger and Heisenberg pictures. In particular, Example 7.8 goes through for arbitrary braided monoidal dagger categories with dagger duals: measurements are completely positive maps from a classical structure to an arbitrary dagger Frobenius structure, and controlled preparations go in the opposite direction.

Lemma 7.21 (CP preserves daggers)**.** *Let $(A, \ , \ )$ and $(B, \ , \ )$ be special dagger Frobenius structures in a monoidal dagger category with dagger duals. If $A \xrightarrow{f} B$ satisfies the CP condition* (7.5)*, then so does $B \xrightarrow{f^\dagger} A$.*

Proof. We show that $f^\dagger$ satisfies the CP condition.



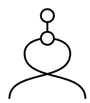
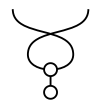
The first two equations use the Frobenius law and unitality. The last equation uses the definition of transpose, and needs X to have a dagger dual. □

We can now prove the closure property stated in the introduction to this chapter: using the CP construction, we do not need to step outside the realm of compact dagger categories. The following theorem summarizes all the structure preserved by the CP construction.

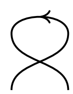
Theorem 7.22 (CP is compact dagger)**.** *If* $\mathbf{C}$ *has the property on the left, then* $\mathrm{CP}[\mathbf{C}]$ *has the property on the right of the following table.*

$\mathbf{C}$	$\mathrm{CP}[\mathbf{C}]$
monoidal dagger category	*category*
braided monoidal dagger category	*monoidal category with right duals*
symmetric monoidal dagger category	*compact category*
compact dagger category	*compact dagger category*

Proof. Combine Proposition 7.19, Lemma 7.20 and Lemma 7.21. All that is left to prove is that if $\mathbf{C}$ is a compact dagger category, then the chosen dualities in $\mathrm{CP}[C]$ are dagger dualities. Using the notation $L \dashv R$ of Lemma 7.20,

 :=  $: L \otimes R \to I$  :=  $: R \otimes L \to I$

satisfy the CP condition, as both are the composition of the swap map and the dagger of a map we have already shown to satisfy the CP condition. The snake equations again come down to the Frobenius law. By definition:

 = 

so in this case L and R are dagger dual objects in $\mathrm{CP}[\mathbf{C}]$. □

Example 7.23. As for examples:

- It follows immediately from Theorems 5.29 and 7.15 that $\mathrm{CP}[\mathbf{FHilb}]$ is the category of finite-dimensional operator algebras and completely positive maps, and that this is a compact dagger category. This is, of course, the category we modeled the CP construction on in the first place. In fact, by Corollary 3.35 and Theorem 5.15 we can even say that $\mathrm{CP}[\mathbf{Hilb}]$ is the same category of finite-dimensional operator algebras and completely positive maps.
- Similarly, Theorem 5.37 and Proposition 7.10 say that $\mathrm{CP}[\mathbf{Rel}]$ is the category of groupoids and inverse-respecting relations, which is a compact dagger category.

7.3 Classical structures

This section considers completely positive maps to and from classical structures. We will see that the subcategory of classical structures and completely positive maps models statistical mechanics, as expected when taking mixed states of classical systems.

Definition 7.24 (The category CP_c)**.** Let $\mathbf{C}$ be a braided monoidal dagger category. The category $\mathrm{CP}_\mathrm{c}[\mathbf{C}]$ has as objects classical structures in $\mathbf{C}$. Its morphisms are completely positive maps.

Again, as before, if $\mathbf{C}$ is compact, then so is $\mathrm{CP}_\mathrm{c}[\mathbf{C}]$. In fact, according to Lemma 7.20, any object in $\mathrm{CP}_\mathrm{c}[\mathbf{C}]$ is self-dual.

As for examples: the next subsection investigates $\mathrm{CP}_\mathrm{c}[\mathbf{FHilb}]$. In the case of **Rel**, completely positive maps between classical structures have no well-known simplification. All we can say is that $\mathrm{CP}_\mathrm{c}[\mathbf{Rel}]$ consists of abelian groupoids and inverse-respecting relations.

Stochastic matrices

If $\mathbf{C}$ models pure state quantum mechanics, and CP[$\mathbf{C}$] mixed state quantum mechanics, then CP$_c$[$\mathbf{C}$] models *statistical mechanics*.

Example 7.25. The category CP$_c$[**FHilb**] is monoidally equivalent to the following: objects are natural numbers, and morphisms $m \to n$ are m-by-n matrices whose entries are nonnegative real numbers. The maps that preserve counits correspond to those matrices whose rows sum up to one, *i.e. stochastic matrices.*

Proof. In **FHilb**, classical structures $(H, \ , \ )$ correspond to a choice of orthonormal basis on H by Corollary 5.31. Hence we may identify linear maps between them with matrices. The positive elements of the classical structure corresponding to the standard basis on $\mathbb{C}^n$ are by definition precisely the vectors whose coordinates are nonnegative real numbers. By Theorem 7.15, a completely positive map $\mathbb{C}^m \xrightarrow{f} \mathbb{C}^n$ must make $f(|i\rangle)$ a positive element of $\mathbb{C}^n$. Combining the last two facts shows that f's matrix has nonnegative real entries $\langle j|f|i\rangle$.

Conversely, any special dagger Frobenius structure H in **FHilb** has an orthonormal basis $|k\rangle$ of positive elements by Theorem 5.29. To verify that $f \otimes \mathrm{id}_H : \mathbb{C}^m \otimes H \to \mathbb{C}^n \otimes H$ is a positive morphism, it suffices to observe that it sends $|i\rangle \otimes |k\rangle$ to the positive element $f(|i\rangle) \otimes |k\rangle$.

The counit of the classical structure $\mathbb{C}^n$ is $(x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n$. So $\mathbb{C}^m \xrightarrow{f} \mathbb{C}^n$ preserves counits when $\sum_{i=1}^n \langle j|f|i\rangle = 1$. □

The previous example is consistent with the morphisms between classical structures we studied in Chapter **??**. Corollary 5.34 showed that comonoid homomorphisms between classical structures correspond to matrices where every column has a single entry one and zeroes otherwise. These are the *deterministic* maps within the stochastic setting of the previous example. Lemma 5.35 showed that these are self-conjugate, which means that their matrix entries are real numbers.

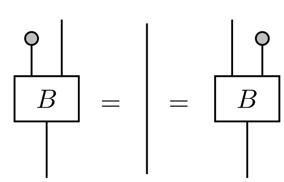
Broadcasting

We now come full circle from showing that compact dagger categories do not support uniform copying and deleting. However, that does not yet guarantee that they model quantum mechanics. Classical mechanics might have copying, and quantum mechanics might not, but statistical mechanics has no copying either. What sets quantum mechanics apart is the fact that *broadcasting* of unknown mixed states is impossible. Before we can get to the precise definition, we have to make sure that there exist *discarding* morphisms $A \to I$ in CP[$\mathbf{C}$].

Lemma 7.26. *Let $(A, \ , \ )$ be a dagger Frobenius structure in a braided monoidal dagger category* $\mathbf{C}$. *Then satisfies the CP condition. If additionally $(A, \ , \ )$ is a classical structure, then satisfies the CP condition.*

Proof. Verifying the CP condition for just comes down to unitality and the fact that the identity is positive. If is commutative, the CP condition for can be rewritten into positive form easily using the noncommutative spider Theorem 5.21. □

Definition 7.27. let $\mathbf{C}$ be a braided monoidal dagger category. A *broadcasting map* for an object $(A, \ , \ )$ of CP[$\mathbf{C}$] is a morphism $A \xrightarrow{B} A \otimes A$ in CP[$\mathbf{C}$] satisfying the following equation:



(7.10)

The object $(A, \ , \ )$ is called *broadcastable* if it allows a broadcasting map.

Notice that the previous definition concerns just a single object, and is therefore much weaker than Definition 4.20. Nevertheless, we can prove it holds for all classical structures.

Lemma 7.28. *Let* $\mathbf{C}$ *be a braided monoidal dagger category. Classical structures are broadcastable objects in* $\mathrm{CP}[\mathbf{C}]$.

Proof. Let $(A, \;, \;)$ be a classical structure. We will show that is a broadcasting map. It clearly satisfies (7.10), so it suffices to show that it is a well-defined morphism in $\mathrm{CP}[\mathbf{C}]$. This follows directly from Lemma 7.26. $\square$

In **FHilb**, the converse to the previous lemma holds; this is the so-called *no-broadcasting theorem*. So a dagger Frobenius structure in **FHilb** is broadcastable if and only if it is a classical structure. However, this is not the case in **Rel**. Call a category *totally disconnected* when its only morphisms are endomorphisms. Totally disconnected groupoids are the extreme opposite of indiscrete ones.

Lemma 7.29. *Broadcastable objects in* $\mathrm{CP}[\mathbf{Rel}]$ *are precisely totally disconnected groupoids.*

Proof. Let $\mathbf{G}$ be a totally disconnected groupoid, and write G for its set of morphisms. We will show that the morphism $G \xrightarrow{B} G \times G$ in **Rel** given by

$$B = \{\big(g, (\mathrm{id}_{\mathrm{dom}(g)}, g)\big) \mid g \in G)\} \cup \{\big(g, (g, \mathrm{id}_{\mathrm{dom}(g)})\big) \mid g \in G\}$$

is a broadcasting map. First of all, B respect inverses because $\mathrm{id}_{\mathrm{dom}(g)} = \mathrm{id}_{\mathrm{dom}(g)^{-1}}$ by total disconnectedness, so B is a well-defined morphism in $\mathrm{CP}[\mathbf{Rel}]$. When interpreted in **Rel**, the broadcastability equation (7.10) reads

$$\{(g,g) \mid g \in G\} = \{(g,h) \mid (g, (\mathrm{id}_C, h)) \in B \text{ for some object } C\} \tag{7.11}$$
$$\{(g,g) \mid g \in G\} = \{(g,h) \mid (g, (h, \mathrm{id}_C)) \in B \text{ for some object } C\}. \tag{7.12}$$

These equations are satisfied by construction of B, and so B is a broadcasting map for $\mathbf{G}$.

Conversely, suppose that a groupoid $\mathbf{G}$ is broadcastable, so that there is a morphism B in **Rel** respecting inverses and satisfying (7.11) and (7.12). Let g be a morphism in $\mathbf{G}$. There is an object C of $\mathbf{G}$ such that $(g, (\mathrm{id}_C, g)) \in B$ by (7.11). Since B respects inverses, then also $(\mathrm{id}_{\mathrm{dom}(g)}, (\mathrm{id}_C, \mathrm{id}_{\mathrm{dom}(g)})) \in B$. But then $C = \mathrm{dom}(g)$ by (7.12). On the other hand, as B respects inverses also $(g^{-1}, (\mathrm{id}_C, g^{-1})) \in B$. Again because B respects inverses then $(\mathrm{id}_{\mathrm{cod}(g)}, (\mathrm{id}_C, \mathrm{id}_{\mathrm{cod}(g)})) \in B$, and so $C = \mathrm{cod}(g)$ by (7.12). Hence $\mathrm{dom}(g) = \mathrm{cod}(g)$, and $\mathbf{G}$ is totally disconnected. $\square$