Categories and Quantum Informatics

Week 7: Complementarity

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Overview

- Incompatible Frobenius structures: mutually unbiased bases
- Deutsch–Jozsa algorithm: prototypical use of complementarity
- Quantum groups: strong complementarity
- Qubit gates: quantum circuits

Idea

- Measure qubit in basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$, then in $\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\}$: probability of either outcome $1/2$.

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- First measurement provides no information about second: Heisenberg's *uncertainty principle*.

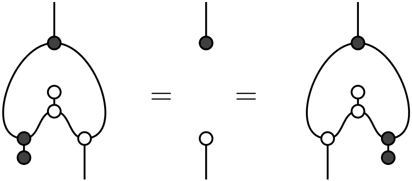
- Orthogonal bases $\{a_i\}$ and $\{b_j\}$ are complementary/unbiased if

$$\langle a_i | b_j \rangle \langle b_j | a_i \rangle = c$$

  for some $c \in \mathbb{C}$.

Complementarity

In braided monoidal dagger category, symmetric dagger Frobenius structures ● and ○ on the same object are complementary if:



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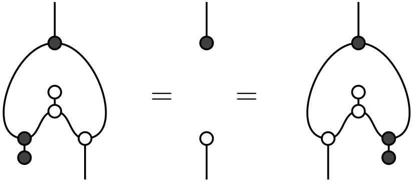
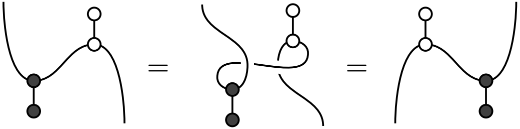


Black and white not obviously interchangeable. But by symmetry:



So could have added two more equalities.

Complementarity in **FHilb**

Commutative dagger Frobenius structures in **FHilb** complementary if and only if they copy complementary bases (with $c = 1$).

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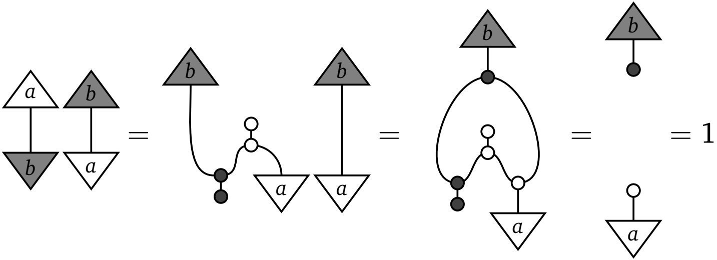
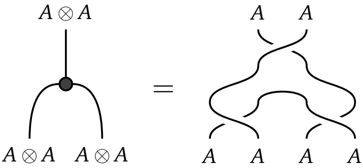
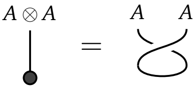
Proof. For all a in white basis, and b in black basis:



Twisted knickers

In compact dagger category, if A is self-dual, the following Frobenius structure on $A \otimes A$ is complementary to pair of pants:





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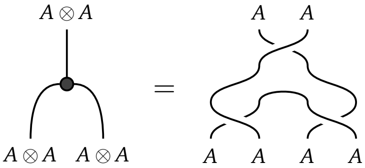

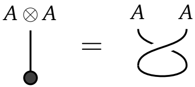

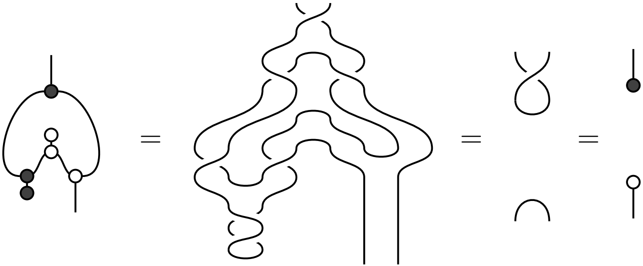

So Frobenius structure on A gives complementary pair on $A \otimes A$.

Pauli basis

Three mutually complementary bases of $\mathbb{C}^2$:

$$\begin{array}{ll}
X \text{ basis} & \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \\
Y \text{ basis} & \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \\
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- Largest family of complementary bases for $\mathbb{C}^2$:
  no four bases all mutually unbiased.

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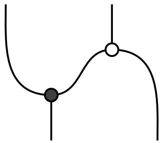
\end{array}$$

- Largest family of complementary bases for $\mathbb{C}^2$: no four bases all mutually unbiased.
- What is the maximum number of mutually complementary bases in a given dimension? Only known for prime power dimensions p^n.

Characterisation

Symmetric dagger Frobenius structures in braided monoidal dagger category complementary if and only if the following is unitary:



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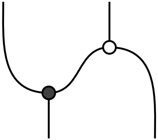
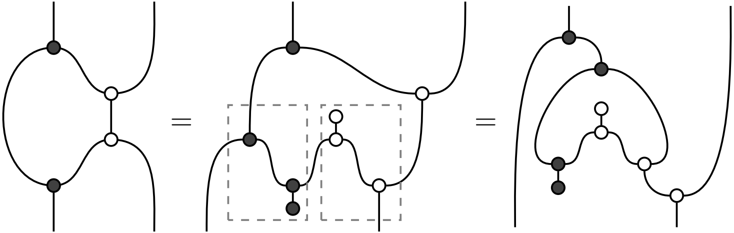


Proof. Compose with adjoint:



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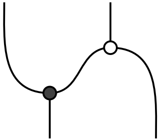

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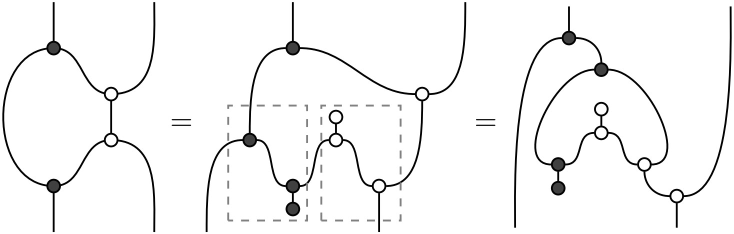

Conversely, if is identity, compose with white counit on top right, black unit on bottom left, to get complementarity.

Complementarity in **Rel**

If G, H are nontrivial groups, these are complementary groupoids:

- objects $g \in G$, morphisms $g \xrightarrow{(g,h)} g$, with $(g, h') \bullet (g, h) = (g, hh')$
- objects $h \in H$, morphisms $h \xrightarrow{(g,h)} h$, with $(g', h) \circ (g, h) = (gh', h)$

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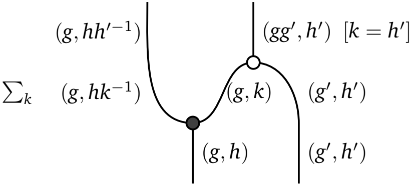
Proof.



Every input related to unique output, so unitary.

Groupoid allows complementary one just when every object has number of outgoing morphisms.

The Deutsch-Jozsa algorithm

Solves certain problem faster than possible classically

- Typical exact quantum decision algorithm (no approximation)
- Problem artificial, but other important algorithms very similar:
  - Shor’s factoring algorithm
  - Grover’s search algorithm
  - the hidden subgroup problem
- ‘All or nothing’ nature makes it categorical

The Deutsch-Jozsa algorithm

Problem:

- Given 2-valued function $A \xrightarrow{f} \{0,1\}$ on a finite set A.
- Constant if takes just a single value on every element of A.
- Balanced if takes value 0 on exactly half the elements of A.
- You are promised that f is either constant or balanced.
  You must decide which.

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Best classical strategy:

- Sample f on $\frac{1}{2}|A| + 1$ elements of A.
  If different values then balanced, otherwise constant.

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How to access f? Can only apply unitary operators...

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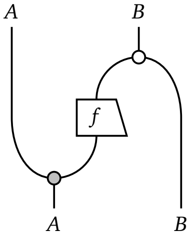
How to access f? Can only apply unitary operators...
Must embed $A \xrightarrow{f} \{0, 1\}$ into an *oracle*.

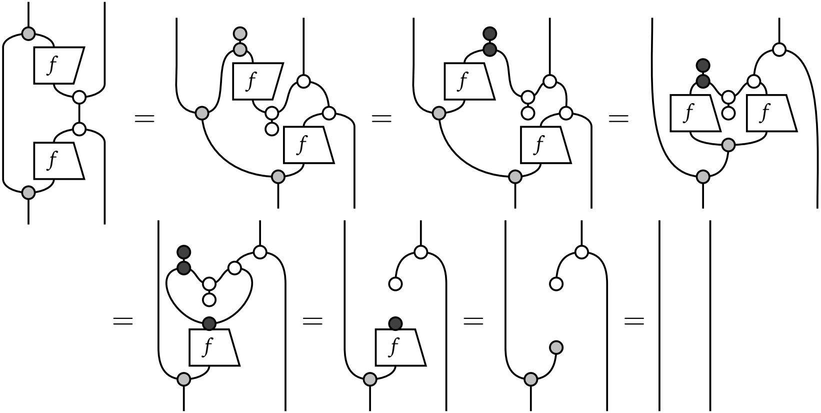
Given Frobenius structures $(A,$ ⅄, ⫰$)$ and $(B,$ ⅄, ⫰$)$ in monoidal dagger category, oracle is morphism $A \xrightarrow{f} B$ making the following unitary:



Where to find oracles

Let $(A,$ ◍$)$, $(B,$ ●$)$ and $(B,$ ○$)$ be symmetric dagger Frobenius.
If ●, ○ complementary, self-conjugate comonoid homomorphism
$(A,$ ◍$) \xrightarrow{f} (B,$ ●$)$ is oracle.

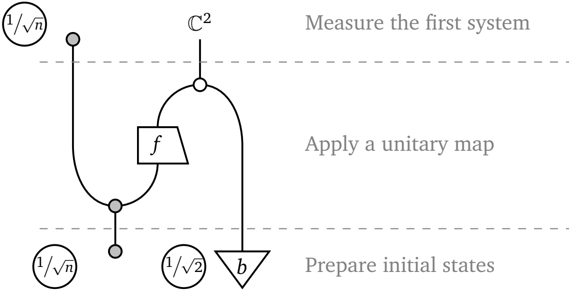
Proof.



The Deutsch-Jozsa algorithm

Let $A \xrightarrow{f} \{0, 1\}$ be given function, and $|A| = n$.
Choose complementary bases ● $= \mathbb{C}^2$, ○ $= \mathbb{C}[\mathbb{Z}_2]$.
Let $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, a copyable state of ○.

The Deutsch–Jozsa algorithm is this morphism:



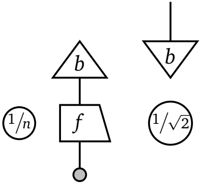
Measure the first system

Apply a unitary map

Prepare initial states

Deutsch-Jozsa simplifies

The Deutsch–Jozsa algorithm simplifies to:



Proof. Duplicate copyable state *b* through white dot, and apply noncommutative spider theorem to cluster of gray dots.

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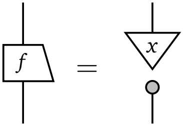

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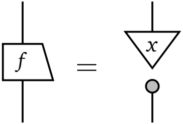
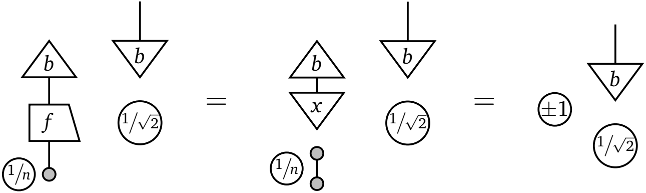


So history is:



This has norm 1, so the history is certain.

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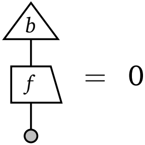

Recall $b = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$.

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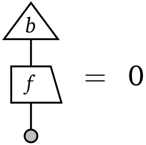

Recall $b = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$. Hence the final history equals 0.

Bialgebras

Complementary classical structures in **FHilb** are mutually unbiased bases. How to build them?

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One standard way: let G be finite group, and consider Hilbert space with basis $\{g \in G\}$, with

⋎: $g \mapsto g \otimes g$ ⫯: $g \mapsto 1$

⋏: $g \otimes h \mapsto gh$ ⫰: $1 \mapsto \sum_{g \in G} g$

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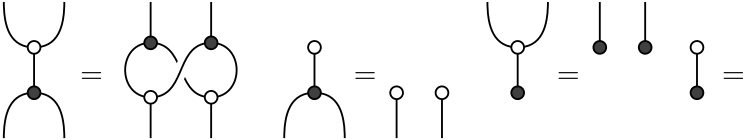
Some nice relationships emerge between ᛉ and ᛦ.

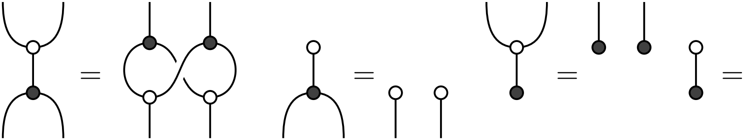
Bialgebras

In a braided monoidal category, a bialgebra consists of a monoid $(A,$ $,$ $)$ and a comonoid $(A,$ $,$ $)$ satisfying:



Bialgebras

In a braided monoidal category, a bialgebra consists of a monoid $(A, ⅄, ⫰)$ and a comonoid $(A, Y, ⫯)$ satisfying:



Example: monoid M is a bialgebra in **Set** and hence in **Rel** and **FHilb**

$$Y\colon m \mapsto (m,m) \qquad ⫯\colon m \mapsto \bullet \qquad ⅄\colon (m,n) \mapsto mn \qquad ⫰\colon \bullet \mapsto 1_M.$$

Frobenius hates bialgebras

In a braided monoidal category, if a monoid $(A,$ $,$ $)$ and comonoid $(A,$ $,$ $)$ form a Frobenius structure and a bialgebra, then $A \simeq I$.

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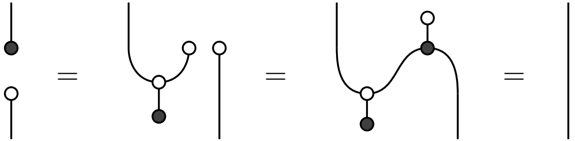
Proof. Will show ● and ○ are inverses. The bialgebra laws already require ○ $\circ$ ● $= \mathrm{id}_I$. For the other composite:



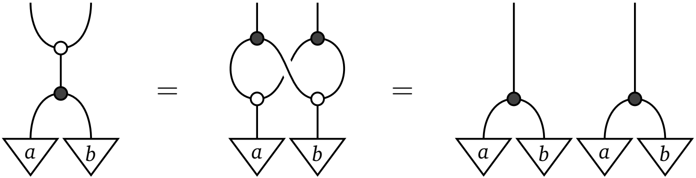
Copyable states

In a braided monoidal category if and form bialgebra, then copyable states for are monoid under .

Copyable states

In a braided monoidal category if ⅄ and Y form bialgebra, then copyable states for Y are monoid under ⅄.

Proof. Associativity is immediate. Unitality comes down to third bialgebra law: ● is copyable for Y. Have to prove well-definedness. Let *a* and *b* be copyable states for Y.



Hence Y-copyable states are indeed closed under ⅄.

Strong complementarity

- Consider $\mathbb{C}^2$ in **FHilb**. Computational basis $\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ gives dagger Frobenius structure ●. Orthogonal basis $\{\begin{pmatrix}e^{i\varphi}\\e^{i\theta}\end{pmatrix}, \begin{pmatrix}e^{i\varphi}\\-e^{i\theta}\end{pmatrix}\}$ gives dagger Frobenius structure ○. Complementary, but only a bialgebra if $\varphi = \theta = 0$.

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- In a braided monoidal dagger category, two dagger symmetric Frobenius structures are strongly complementary when they are complementary, and also form a bialgebra.

Strong complementarity in **FHilb**

In **FHilb**, strongly complementary symmetric dagger Frobenius structures, one of which is commutative, correspond to finite groups.

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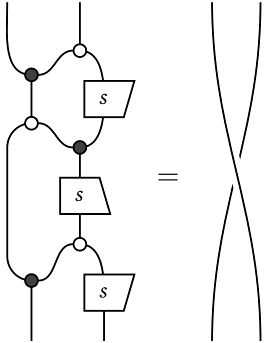
Proof.

- Given strongly complementary symmetric dagger Frobenius structures, the states that are self-conjugate, copyable and deletable for (,) form a group under .
- By the classification theorem for commutative dagger Frobenius structures, there is an entire basis of such states for .

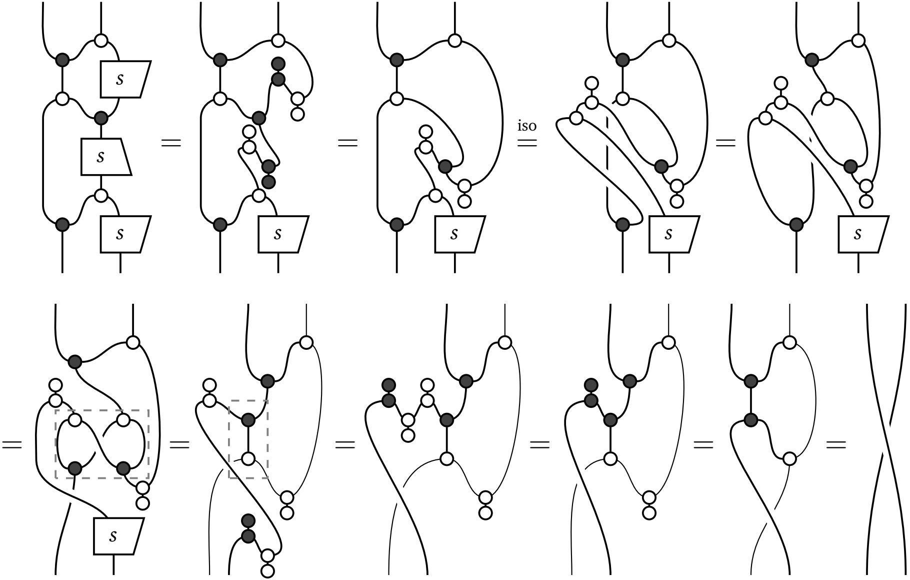
Qubit gates

In a braided monoidal dagger category, let (⅄, ⫯) and (Y, ⫰) be complementary classical structures with antipode *s*. Then the first bialgebra law holds if and only if:



Qubit gates

Proof.



Qubit gates in **FHilb**

Fix A to be qubit $\mathbb{C}^2$; let (⋏, •) copy computational basis $\{|0\rangle, |1\rangle\}$, and (⋎, ∘) copy the X basis. The three antipodes s become identities.

The three unitaries reduce to three CNOT gates:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

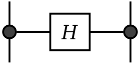
These two classical structures are transported into each other by Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{H}$$

Controlled Z

The CZ gate in **FHilb** can be defined as follows.

CZ =



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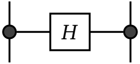
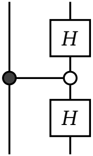


Proof. Rewrite as:

CZ =



Hence

$$\mathrm{CZ} = (\mathrm{id} \otimes H) \circ \mathrm{CNOT} \circ (\mathrm{id} \otimes H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

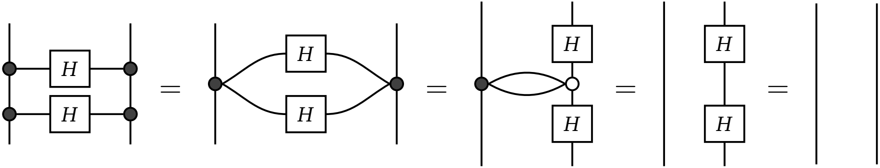
Controlled Z

If $(A, ⋏)$ and $(A, ⋎)$ complementary classical structures in braided monoidal dagger category, and $A \xrightarrow{H} A$ satisfies $H \circ H = \mathrm{id}_A$, then CZ makes sense and satisfies $\mathrm{CZ} \circ \mathrm{CZ} = \mathrm{id}$.

Controlled Z

If $(A, ⅄)$ and (A, Y) complementary classical structures in braided monoidal dagger category, and $A \xrightarrow{H} A$ satisfies $H \circ H = \mathrm{id}_A$, then CZ makes sense and satisfies $\mathrm{CZ} \circ \mathrm{CZ} = \mathrm{id}$.

Proof.



Measurement-based computing

Single-qubit unitaries can be implemented via Euler angles: unitary $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$ allows phases φ, ψ, θ with $u = Z_\theta \circ X_\psi \circ Z_\varphi$, where Z_θ is rotation in Z basis over angle θ, and X_φ in X basis over angle φ.

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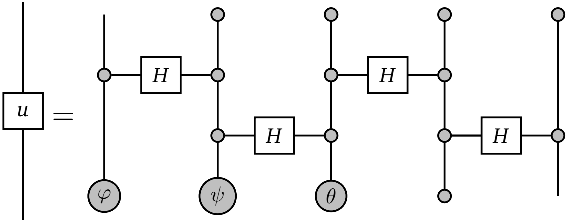
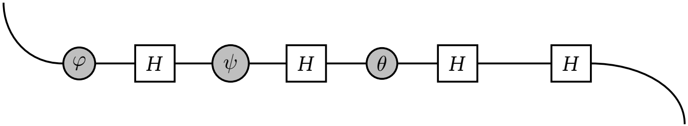
If unitary $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$ in **FHilb** has Euler angles φ, ψ, θ, then:

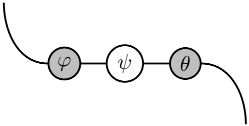


Measurement-based computing

Proof. Use phased spider theorem to reduce to:



But by transport lemma, this is just:



which equals u, by definition of the Euler angles.

Summary

- Incompatible Frobenius structures: mutually unbiased bases
- Deutsch-Jozsa algorithm: prototypical use of complementarity
- Quantum groups: strong complementarity
- Qubit gates: use in quantum circuits