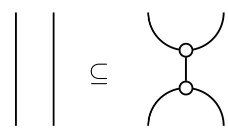
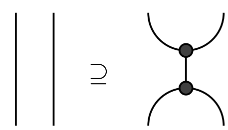
Categories and Quantum Informatics exercise sheet 7: Complementarity

Exercise 5.1. Let $(G, \circ)$ and $(G, \bullet)$ be two complementary groupoids (see Proposition 6.9).

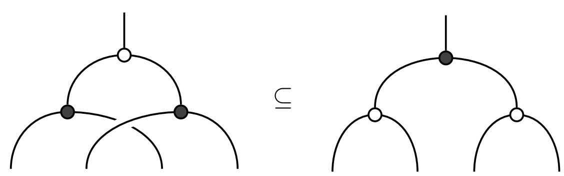
(a) Assume that $(G, \circ)$ is a group. Show that:



(b) Assume that $(G, \circ)$ is a group. Show that:



(c) Assume that $(G, \circ)$ is a group and that the corresponding Frobenius structures in **Rel** form a bialgebra. Show that:



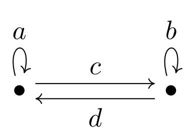
Exercise 5.2. Let A be a set with a prime number of elements. Show that pairs of complementary Frobenius structures on A in **Rel** correspond to groups whose underlying set is A.

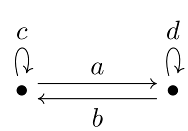
Exercise 5.3. Consider a special dagger Frobenius structure in **Rel** corresponding to a groupoid $\mathbf{G}$.

(a) Show that nonzero copyable states correspond to endohomsets $\mathbf{G}(A, A)$ of $\mathbf{G}$ that are isolated in the sense that $\mathbf{G}(A, B) = \emptyset$ for each object B in $\mathbf{G}$ different from A.

(b) Show that unbiased states of $\mathbf{G}$ correspond to sets containing exactly one morphism into each object of $\mathbf{G}$ and exactly one morphism out of each object of $\mathbf{G}$.

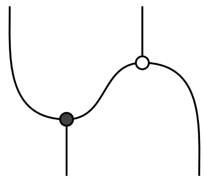
(c) Consider the following two groupoids on the morphism set $\{a, b, c, d\}$.





Show that copyable states for one are unbiased for the other, but that they are not complementary. Conclude that the converse of Proposition 6.11 is false.

Exercise 5.4. A *Latin square* is an n-by-n matrix L with entries from $\{1, \ldots, n\}$, with each $i = 1, \ldots, n$ appearing exactly once in each row and each column. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for $\mathbb{C}^n$. Define : $\mathbb{C}^n \longrightarrow \mathbb{C}^n \otimes \mathbb{C}^n$ by $e_i \mapsto e_i \otimes e_i$, and : $\mathbb{C}^n \otimes \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by $e_i \otimes e_j \mapsto e_{L_{ij}}$. Show that the composite



is unitary. Note that  need not be associative or unital.

Exercise 5.5. This exercise is about *property* versus *structure*. Suppose that a category **C** has products. Show that any monoid in **C** has a unique bialgebra structure.