

Categories and Quantum Informatics

Week 6: Frobenius structures

Chris Heunen



THE UNIVERSITY *of* EDINBURGH
informatics

Overview

- ▶ Frobenius structure: interacting co/monoid, self-duality
- ▶ Normal forms: coherence theorem
- ▶ Frobenius law: coherence between dagger and closure
- ▶ Classification: in **FHilb** and **Rel**
- ▶ Phases: unitary operators

Idea

Orthonormal basis $\{e_i\}$ for H in **FHilb** gives comonoid $\wp: e_i \mapsto e_i \otimes e_i$.
Its adjoint \wp^* is **comparison**: $e_i \otimes e_i \mapsto e_i$ and $e_i \otimes e_j \mapsto 0$ if $i \neq j$.

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These cooperate:

The diagrammatic equation illustrates the relationship between the comultiplication map \wp and the comparison map \wp^* . On the left, a single input line splits into two lines that then merge back into one. The top merge node is a circle with a dot, representing \wp . The bottom split node is a circle with a dot, representing \wp^* . The two resulting lines end in triangles labeled e_i and e_j . This is equated to a piecewise definition in square brackets: the top part shows two triangles labeled e_i and e_j with the text "if $i = j$ ", and the bottom part shows the number "0" with the text "if $i \neq j$ ". This is further equated to a diagram where the top merge node is a circle with a dot, and the two lines from the bottom split node go directly to the triangles labeled e_i and e_j .

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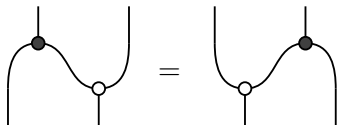
These cooperate:

The diagrammatic equation shows the Frobenius law for the comonoid \wp . On the left, a wire enters from the top, passes through a multiplication node (a circle with a dot), then splits into two wires that pass through comultiplication nodes (triangles pointing down) labeled e_i and e_j . This is equal to a piecewise definition in square brackets: the top part shows two comultiplication nodes e_i and e_j with the condition "if $i = j$ ", and the bottom part shows a zero 0 with the condition "if $i \neq j$ ". This is equal to the right-hand diagram, where the wire enters from the top, passes through a comultiplication node e_i , then a multiplication node, and finally splits into two wires that pass through comultiplication nodes e_i and e_j .

This monoid/comonoid interaction is called the **Frobenius law**.

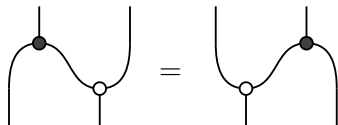
Frobenius structures

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Frobenius structures

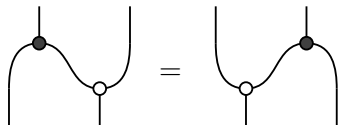
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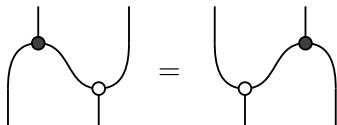
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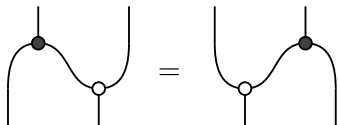
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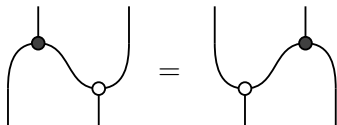
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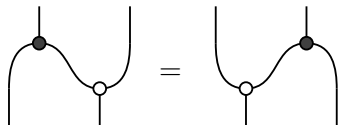
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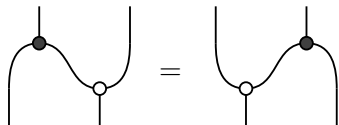
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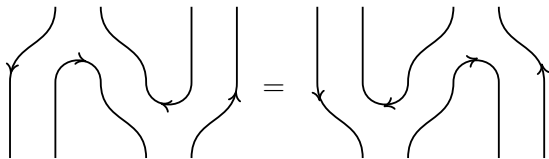
Pair of pants

In a dagger monoidal category, if $A \dashv A^*$, the pair of pants monoid $A^* \otimes A$ carries a dagger Frobenius structure.

Pair of pants

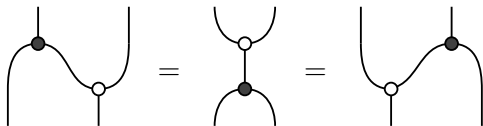
In a dagger monoidal category, if $A \dashv A^*$, the pair of pants monoid $A^* \otimes A$ carries a dagger Frobenius structure.

Proof.



Extended Frobenius law

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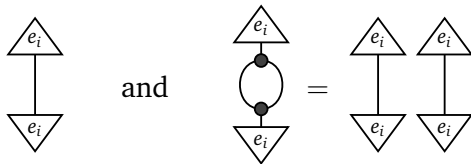
A diagrammatic equation showing three equivalent configurations of a Frobenius structure. The first configuration consists of a vertical line on the left that meets a black dot (multiplication) on a curve that then goes down to a white dot (comultiplication) on another curve, which finally meets a vertical line on the right. The second configuration is a vertical line that splits into two curves at a white dot (comultiplication), which then merge back into a single vertical line at a black dot (multiplication). The third configuration is a vertical line on the left that meets a white dot (comultiplication) on a curve that then goes up to a black dot (multiplication) on another curve, which finally meets a vertical line on the right. All three configurations are connected by equals signs.

Proof.

A sequence of diagrammatic transformations proving the extended Frobenius law. The proof starts with the Frobenius structure from the previous block. It then introduces a new white dot (comultiplication) on the left side of the diagram. Through a series of moves, this dot is moved to the top of the left vertical line, then to the top of the left curve, and finally to the top of the right curve. Each step is shown with an equals sign. The final configuration shows the white dot at the top of the right curve, with a vertical line extending upwards from it.

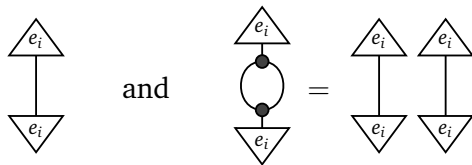
Speciality

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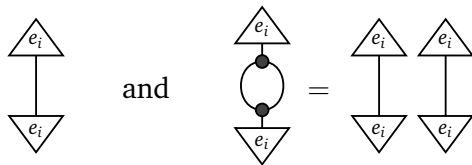
If \mathcal{V} copies orthogonal basis $\{e_i\}$, can find (squared) norm of e_i :



So can characterize orthonormality via Frobenius structure.

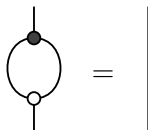
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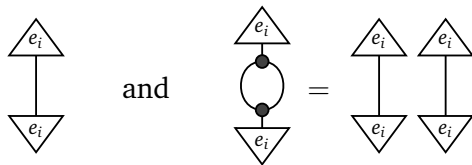
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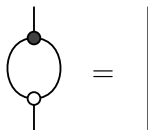
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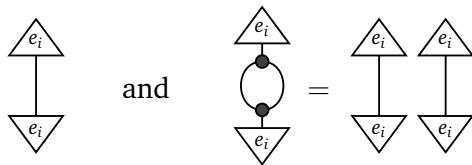
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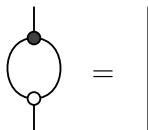
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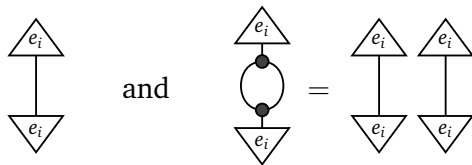


Examples:

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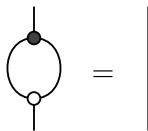
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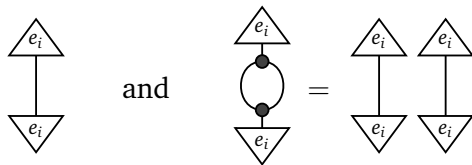


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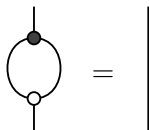
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- ▶ Groupoid Frobenius structure in **Rel** is always special

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- ▶ Speciality and Frobenius law imply (co)associativity
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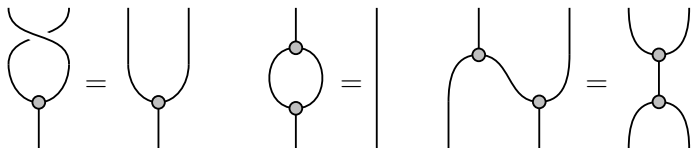
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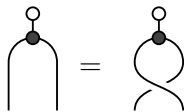
To check that (A, μ, ν) is classical structure, only need:



Symmetry

Pair of pants hardly ever commutative. However:

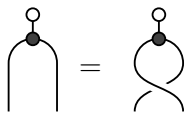
A Frobenius structure is **symmetric** when:



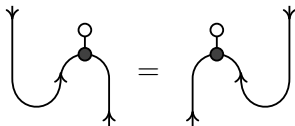
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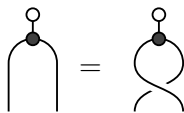
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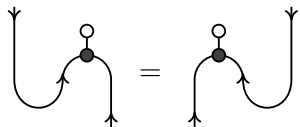
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Examples:

- ▶ Pair of pants: in **FHilb** this says $\text{Tr}(ab) = \text{Tr}(ba)$
- ▶ Group algebras: inverses in groups are two-sided inverses
- ▶ Groupoid Frobenius structure: inverses are two-sided

Self-duality

If $(A, \psi, \varphi, \mu, \nu)$ Frobenius structure in monoidal category, then $A \dashv A$ is self-dual with:

$$\begin{array}{ccc} \begin{array}{c} A \quad A \\ \cup \end{array} & = & \begin{array}{c} A \quad A \\ \cup \\ \circ \\ \bullet \end{array} \end{array} \qquad \begin{array}{ccc} \begin{array}{c} \cap \\ A \quad A \end{array} & = & \begin{array}{c} \circ \\ \bullet \\ \cap \\ A \quad A \end{array} \end{array}$$

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Nondegenerate forms

Monoid $(A, \blacktriangleleft, \blacktriangleright)$ forms Frobenius structure with comonoid (A, ∇, φ) iff allows **nondegenerate form**: map $\varphi: A \rightarrow I$ with



part of self-duality $A \dashv A$.

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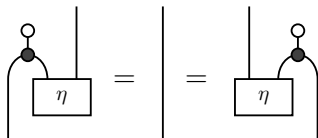
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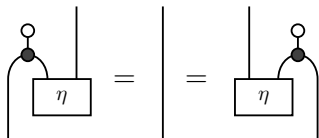
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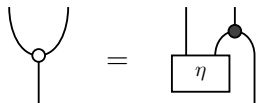
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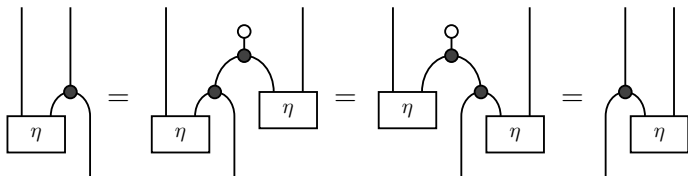
Define comultiplication as:



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Proof (continued.)

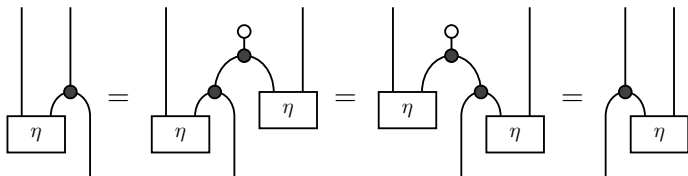
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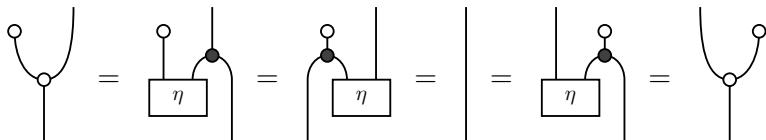
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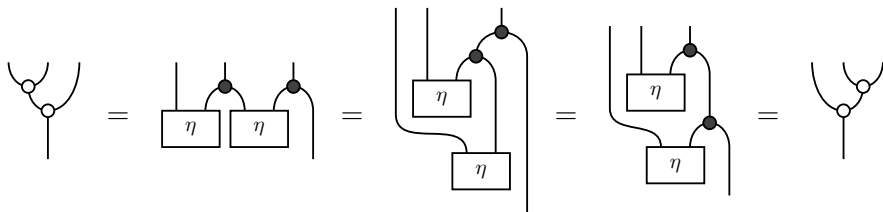
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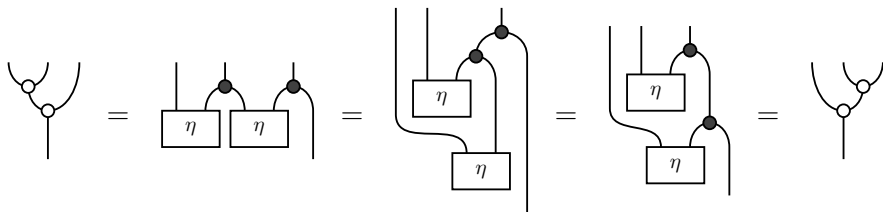
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Frobenius law:



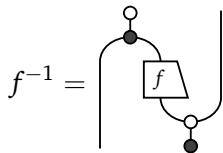
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$$f^{-1} = \begin{array}{c} \circ \\ | \\ \bullet \\ \left. \begin{array}{l} \curvearrowright \\ \square \text{ } f \\ \curvearrowleft \\ \circ \\ | \\ \bullet \end{array} \right\} \end{array}$$

Indeed:

$$\begin{array}{c} \circ \\ | \\ \bullet \\ \left. \begin{array}{l} \curvearrowright \\ \square \text{ } f \\ \curvearrowleft \\ \circ \\ | \\ \bullet \end{array} \right\} \end{array} = \begin{array}{c} \circ \\ | \\ \bullet \\ \left. \begin{array}{l} \curvearrowright \\ \bullet \\ \square \text{ } f \\ \bullet \\ \curvearrowleft \\ \circ \\ | \\ \bullet \end{array} \right\} \end{array} = \begin{array}{c} \circ \\ | \\ \bullet \\ \left. \begin{array}{l} \curvearrowright \\ \bullet \\ \bullet \\ \curvearrowleft \\ \circ \\ | \\ \bullet \end{array} \right\} \end{array} = \begin{array}{c} | \end{array}$$

Normal forms

Two ways to think about graphical calculus:

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merely shorthand to write down e.g. linear map;
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can be manipulated by replacing equal parts.

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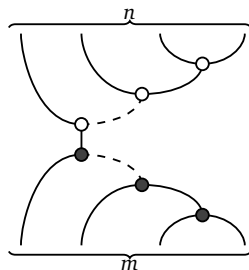
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Unique way to copy (\curvearrowright), discard (\circlearrowleft), fuse (\curvearrowright), create (\circlearrowleft) data!

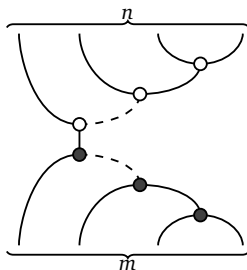
Spider theorem

Let $(A, \mu, \nu, \psi, \phi)$ be a special Frobenius structure. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built out of finitely many pieces μ, ν, ψ, ϕ , and id , using \circ and \otimes , equals:



Spider theorem





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Proof. Induction on the number of dots.







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Proof. (continued.)

Base case. Trivial, as the diagram must be one of , , , , φ .

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


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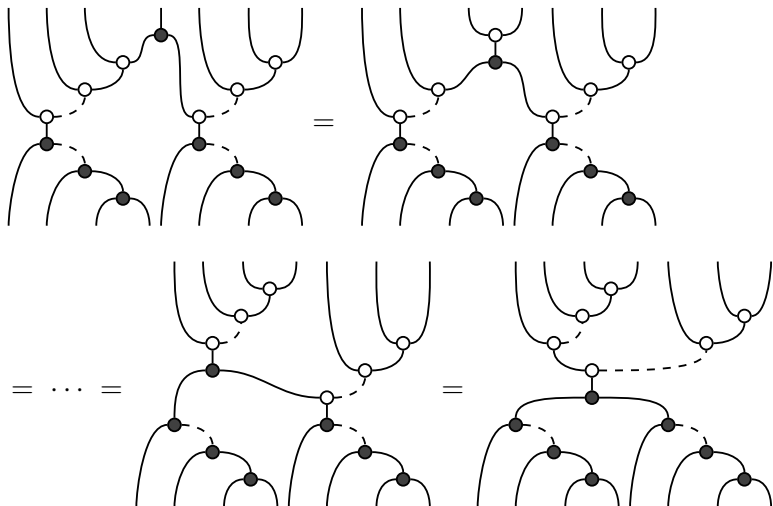
Is the diagram underneath the \curvearrowright connected?

If so, use coassociativity and speciality.

Spider theorem

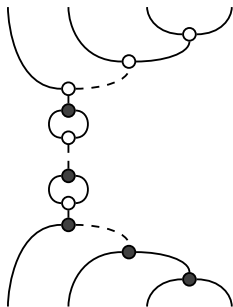
Proof. (continued.)

Suppose instead the rest of the diagram is disconnected:



More spider theorems

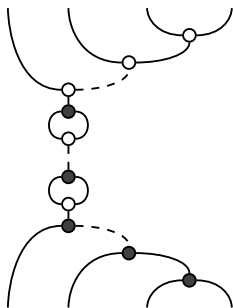
In a monoidal category, let $(A, \mu, \nu, \psi, \varphi)$ be a Frobenius structure. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built out of finitely many pieces μ, ν, ψ, φ , and id , using \circ and \otimes , equals $(*)$.



(*)

More spider theorems

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$(*)$

In a symmetric monoidal category, let $(A, \mu, \nu, \varphi, \psi)$ be a commutative Frobenius structure. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built out of finitely many pieces $\mu, \nu, \varphi, \psi, \text{id}, \times$, using \circ and \otimes , equals $(*)$.

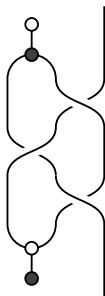
No braided spider theorem

In a braided non-symmetric monoidal category, there is no normal form for commutative Frobenius algebras.

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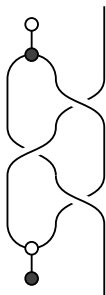
Proof. Regard the following diagram as a piece of string on which an overhand knot is tied:



No braided spider theorem

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The Frobenius algebra axioms induce homotopy equivalences ('deformations') of the corresponding graph. Such moves cannot untie the knot.

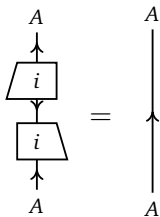
Involutive monoids

If (A, m, u) is monoid, so is (A^*, m_*, u_*) .

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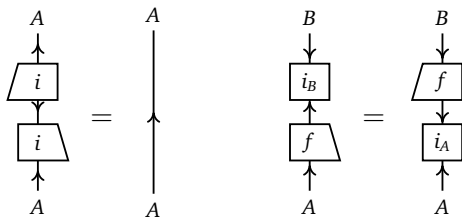
An **involution** for a monoid (A, \cdot, \circ) is a monoid homomorphism $A \xrightarrow{i} A^*$ satisfying $i_* \circ i = \text{id}_A$.



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A **morphism of involutive monoids** is monoid homomorphism $A \xrightarrow{f} B$ satisfying $i_B \circ f = f_* \circ i_A$.

Example involutive monoids

- ▶ **Matrix algebra.** \mathbb{M}_n is an involutive monoid in **FHilb**.
Opposite monoid \mathbb{M}_n^* : multiplication ab in \mathbb{M}_n^* is ba in \mathbb{M}_n .
Canonical involution $\mathbb{M}_n \rightarrow \mathbb{M}_n^*$ given by $f \mapsto f^\dagger$.

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- ▶ **Pair of pants.** $A^* \otimes A$ involutive in a dagger pivotal category.
Identity map as involution, because of conventions:

$$\left(\begin{array}{c} \swarrow \quad \searrow \\ \cap \end{array} \right)_* = \left(\begin{array}{c} \text{U-shaped diagram with arrows} \end{array} \right)^\dagger = \begin{array}{c} \swarrow \quad \searrow \\ \cap \end{array}$$

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Identity map as involution, because of conventions:

The diagram shows the involution of a pair of pants. On the left, a pair of pants with two input wires on the left and one output wire on the right is enclosed in large parentheses with a subscript asterisk. This is equal to the dagger of a pair of pants with one input wire on the left and two output wires on the right, which is shown in large parentheses with a dagger symbol. This is finally equal to the original pair of pants with two inputs on the left and one output on the right.

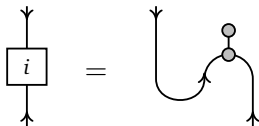
- ▶ **Groupoids.** \mathbf{G} in **Rel** is involutive.
Opposite monoid: induced by opposite groupoid \mathbf{G}^{op}

The diagram shows the involution of a groupoid element. On the left, a morphism is represented by two vertical lines on the left that merge into one vertical line on the right, with a small grey dot at the junction. This is equal to the inverse morphism, represented by one vertical line on the left that splits into two vertical lines on the right, with a small grey dot at the junction.

Canonical involution $G \rightarrow G^*$ given by $g \sim g^{-1}$.

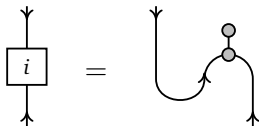
Frobenius law from way of the dagger

Monoid (A, \circ, \circ) is dagger Frobenius if and only if i is involution:



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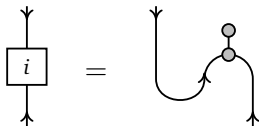
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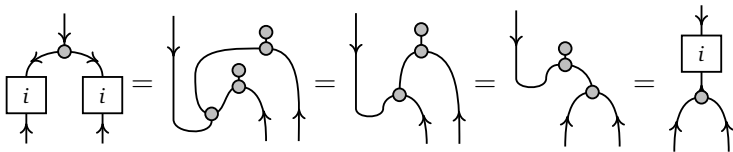
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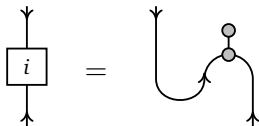
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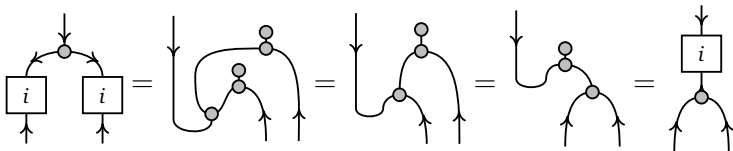
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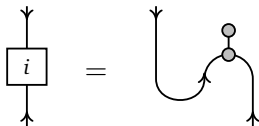
► i preserves multiplication:



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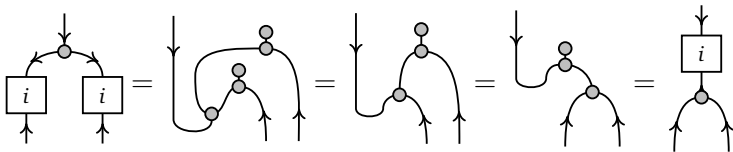
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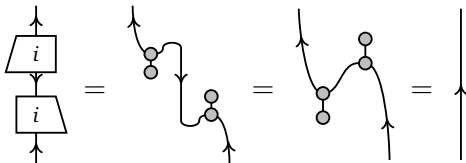
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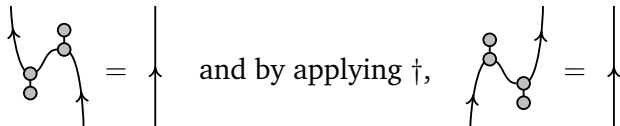
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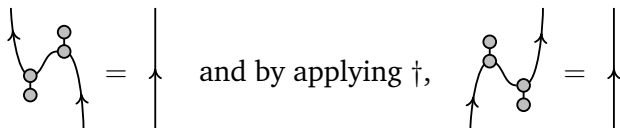
Proof. (continued.) Conversely, suppose $i_* \circ i = \text{id}$. Then:



The diagram shows two equations. The first equation shows a diagram with three nodes and two lines. The left line starts at the top, goes down to a node, then up to a second node, then down to a third node. The right line starts at the top, goes down to a node, then up to a second node, then down to a third node. The two lines are connected at the top and bottom nodes. This diagram is equal to a single vertical line with an upward arrow. The second equation is identical to the first, but the lines are swapped. This diagram is also equal to a single vertical line with an upward arrow. The text "and by applying †," is placed between the two equations.

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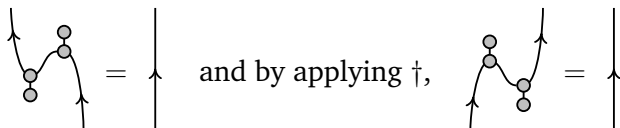


The diagram shows two equations. The first equation shows a diagram with three nodes and two lines on the left, which is equal to a single vertical line with an upward arrow on the right. The second equation shows a diagram with three nodes and two lines on the right, which is equal to a single vertical line with an upward arrow on the left. The text "and by applying †," is placed between the two equations.

So we have a Frobenius structure, defined by a nondegenerate form.
Is it a dagger Frobenius structure?

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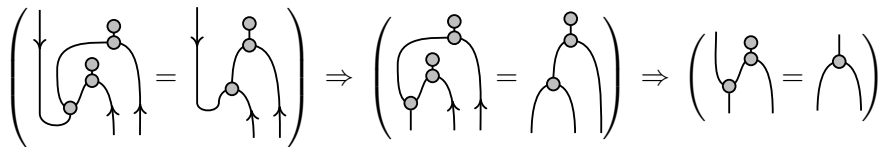


The diagram shows two equations. The first equation shows a diagram with three nodes and four strands (two on the left, two on the right) equal to a single vertical strand. The second equation, preceded by "and by applying †", shows a similar diagram with the strands swapped, also equal to a single vertical strand.

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Is it a dagger Frobenius structure?

The condition that i preserves multiplication gives:



The diagram shows a sequence of three equations connected by double arrows. The first equation shows a complex diagram with four nodes and four strands equal to a simpler diagram with three nodes and four strands. The second equation shows a diagram with four nodes and four strands equal to a diagram with three nodes and four strands. The third equation shows a diagram with three nodes and four strands equal to a diagram with two nodes and four strands.

So the form definition gives rise to the correct comultiplication.

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Theorem: special dagger Frobenius structures in **FHilb** are of the form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$.

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Corollary: classical structure in **FHilb** copy orthonormal bases

Proof: must be of form $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

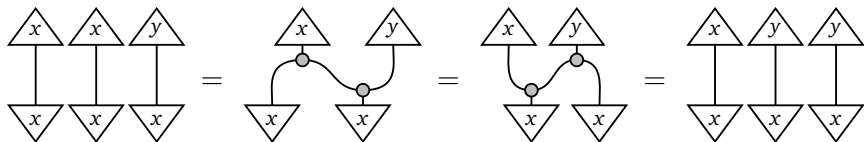
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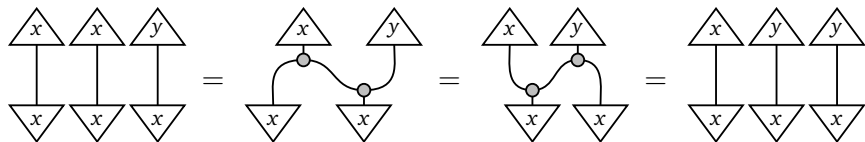
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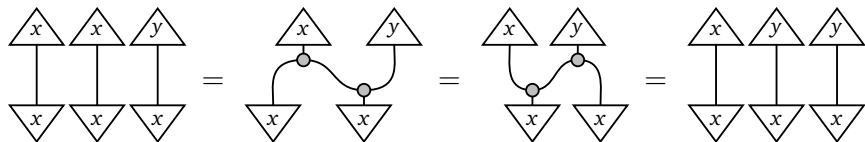


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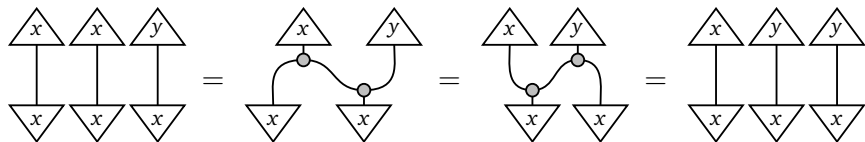
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If $\langle x|y \rangle \neq 0$, this implies $\langle x|x \rangle = \langle x|y \rangle$. Similarly $\langle y|x \rangle = \langle y|y \rangle$.

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Hence $\langle x - y|x - y \rangle = \langle x|x \rangle - \langle x|y \rangle - \langle y|x \rangle + \langle y|y \rangle = 0$, so $x = y$.

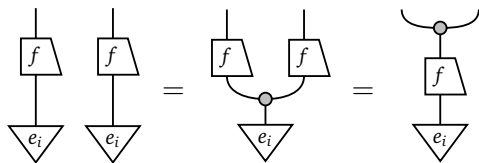
Orthogonal bases and morphisms

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Proof. Suffices to see about basis of copyable states $\{e_i\}$.



Hence $f(e_i)$ copyable.

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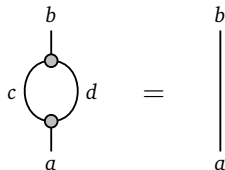
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Classification in **Rel**

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Proof. Write $A \times A \xrightarrow{M} A$ for multiplication, $U \subseteq A$ for unit.

M is single-valued: by speciality $a(M \circ M^\dagger)b$ iff $a = b$:



So: if $(c, d)Ma$ and $(c, d)Mb$, must have $a = b$.

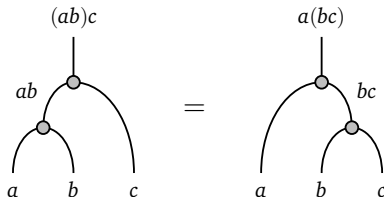
May simply write ab for unique c with $(a, b)Mc$.

Remember: ab not always defined!

Classification in Rel

Proof. (continued)

Associativity:



So ab and $(ab)c$ defined exactly when bc and $a(bc)$ are defined, and then $(ab)c = a(bc)$.

Classification in Rel

Proof. (continued)

Unitality: for units $x, y \in U$

$$\begin{array}{c} b \\ | \\ \bullet \\ / \quad \backslash \\ x \quad \quad a \\ \backslash \quad / \\ \bullet \\ | \\ a \end{array} = \begin{array}{c} b \\ | \\ a \end{array} = \begin{array}{c} b \\ | \\ \bullet \\ / \quad \backslash \\ a \quad \quad y \\ \backslash \quad / \\ \bullet \\ | \\ a \end{array}$$

So: a, b allow $x \in U$ with $xa = b$ iff $a = b$.

And: a, b allow $y \in U$ with $ay = b$ iff $a = b$.

If $z \in U$ then $xz = x$ for some $x \in U$. But then $x = z$!

Units idempotent; multiplication of different ones undefined.

If $xa = a = x'a$, then $a = xa = x(x'a) = (xx')a$, so $x = x'$.

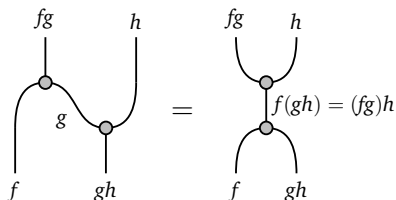
So every element has unique left/right identity.

Classification in Rel

Proof. (continued)

Category: U set of objects, A set of morphisms.

If fg defined and gh defined, want $(fg)h = f(gh)$ defined too:

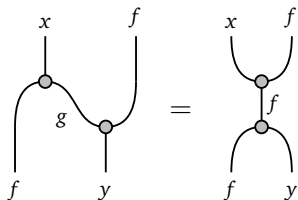


If fg and gh defined then LHS defined, so RHS defined too.

Classification in Rel

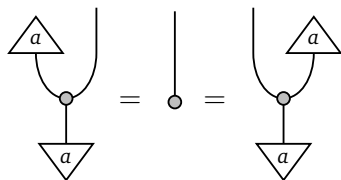
Proof. (continued)

Inverses: for $f \in A$ with left unit x and right unit y :



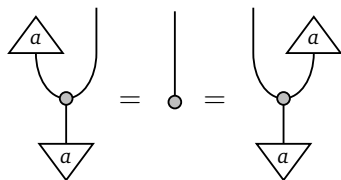
Phases

Let $(A, \otimes, \circlearrowleft, \circlearrowright)$ be Frobenius structure in a monoidal dagger category.
State $I \xrightarrow{a} A$ is called **phase** when:

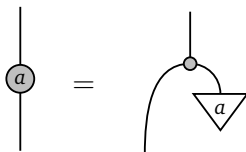


Phases

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Its **(right) phase shift** is the following morphism $A \rightarrow A$:



Example phases

- ▶ For classical structure in **FHilb** copying basis $\{e_i\}$, vector $a = a_1e_1 + \cdots a_n e_n$ is phase iff each a_i on unit circle: $|a_i|^2 = 1$.

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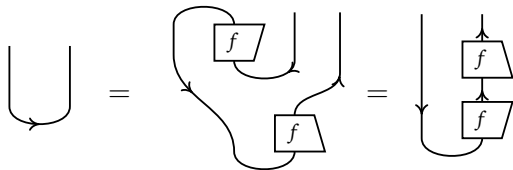
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Proof. The name of an morphism $A \xrightarrow{f} A$ is a phase when:



But this means $f \circ f^\dagger = \text{id}_A$; similarly $f^\dagger \circ f = \text{id}_A$.

Example phases

- ▶ Phases of Frobenius structure \mathbb{M}_n in **FHilb** form set $U(n)$ of n -by- n unitary matrices. Hence phases of $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ range over $U(k_1) \times \cdots \times U(k_n)$.

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- ▶ The phases of a Frobenius structure in **Rel** induced by a group G are elements of that group G itself.

Proof. For a subset $a \subseteq G$, equation defining phases reads

$$\{g^{-1}h \mid g, h \in a\} = \{1_G\} = \{hg^{-1} \mid g, h \in a\}.$$

So if $g \in G$, then $a = \{g\}$ is a phase. But if a contains distinct elements $g \neq h$ of G , cannot be phase. Similarly, $a = \emptyset$ not phase. Hence a phase precisely when singleton $\{g\}$.

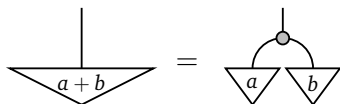
Phase group

In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit \circ and:

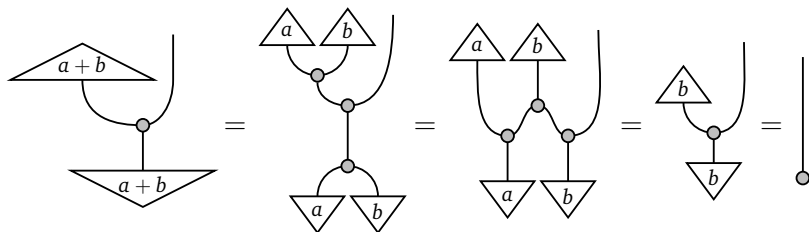
The diagram shows an equality between two expressions. On the left is a dagger Frobenius structure with a phase: a vertical line enters from the top, passes through a small grey circle, and then enters a downward-pointing triangle labeled $a+b$. On the right is the same structure decomposed: a vertical line enters from the top, passes through a small grey circle, and then splits into two curved lines that enter two separate downward-pointing triangles labeled a and b . An equals sign is placed between the two diagrams.

Phase group

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$$\text{Diagram 1} = \text{Diagram 2}$$

Proof. This is again a well-defined phase:


$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} = \text{Diagram 5}$$

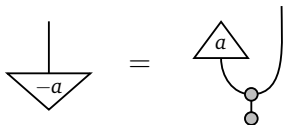
The flipped equation follows similarly.

Associativity is clear, hence phases form a monoid.

Phase group

Proof. (continued)

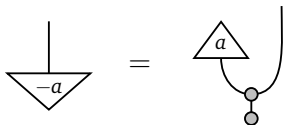
Left-inverse of phase a is:



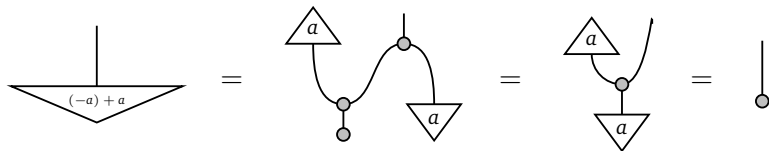
Phase group

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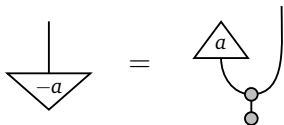
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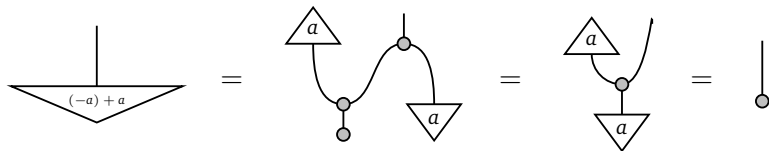
Phase group

Proof. (continued)

Left-inverse of phase a is:



Left-inverse of a is $-a$:



Similarly there is right-inverse. But in monoids, left and right inverses are equal: $l = l(xr) = (lx)r = r$.



Example phase groups

- ▶ In **FHilb**, the phase group for the pair of pants Frobenius structure is the unitary group.

Example phase groups

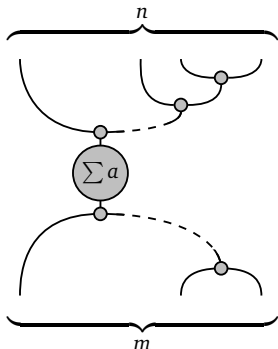
- ▶ In **FHilb**, the phase group for the pair of pants Frobenius structure is the unitary group.
- ▶ Phase addition in the Frobenius structure $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ in **FHilb** is entrywise multiplication in $U(k_1) \times \cdots \times U(k_n)$.
In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.

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In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.
- ▶ In **Rel**, the phase group induced by a group G is the group itself.

Phased spider theorem

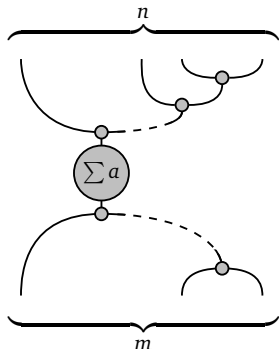
Let $(A, \multimap, \circlearrowleft, \circlearrowright)$ be classical structure in braided monoidal dagger category. Any connected morphism $A^{\otimes m} \rightarrow A^{\otimes n}$ built of finitely many $\multimap, \circlearrowleft, \circlearrowright, \text{id}, \sigma$ and phases using \circ, \otimes , and \dagger , equals



where a ranges over all the phases used in the diagram.

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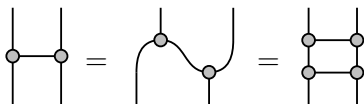
where a ranges over all the phases used in the diagram.

Proof. Use braidings to have all phases dangle at the bottom. Apply Spider Theorem. Use phase addition to reduce to single phase $\sum a$ on bottom right. Apply Spider Theorem again. \square

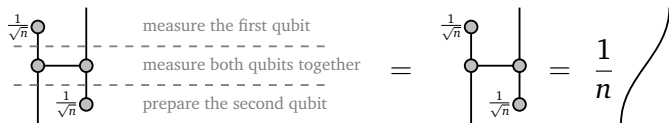
State transfer

State transfer protocol: transfer state of Hilbert space H from one system to another, with success probability $1/\dim(H)^2$.

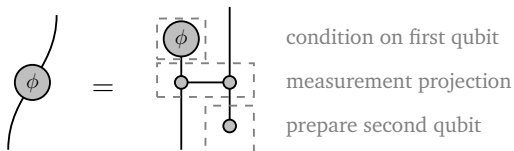
May be lax in drawing, e.g. projection $H \otimes H \rightarrow H \otimes H$:



The procedure looks like this:



Extra challenge: apply phase gate while transferring state



Summary

- ▶ Frobenius structures: interacting co/monoid, self-duality
- ▶ Normal forms: spider theorems
- ▶ Frobenius law: justified by coherence
- ▶ Classification: matrix algebras, bases, groupoids
- ▶ Phases: unitary operators, state transfer

Next week: interaction between two Frobenius structures