Categories and Quantum Informatics

Week 6: Frobenius structures

Chris Heunen

THE UNIVERSITY of EDINBURGH
informatics

Overview

- Frobenius structure: interacting co/monoid, self-duality
- Normal forms: coherence theorem
- Frobenius law: coherence between dagger and closure
- Classification: in **FHilb** and **Rel**
- Phases: unitary operators

Idea

Orthonormal basis $\{e_i\}$ for H in **FHilb** gives comonoid : $e_i \mapsto e_i \otimes e_i$.
Its adjoint is comparison: $e_i \otimes e_i \mapsto e_i$ and $e_i \otimes e_j \mapsto 0$ if $i \neq j$.

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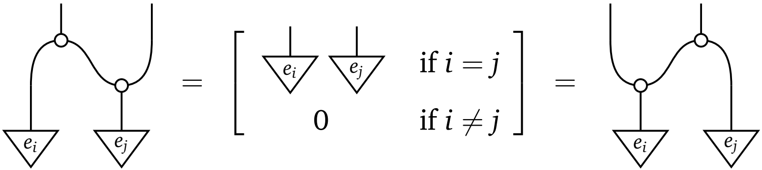

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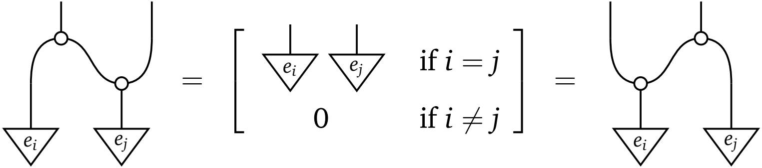
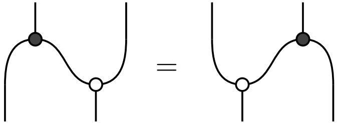


This monoid/comonoid interaction is called the Frobenius law.

Frobenius structures

In a monoidal category, a Frobenius structure is a comonoid $(A, ,)$ and monoid $(A, ,)$ satisfying the Frobenius law:



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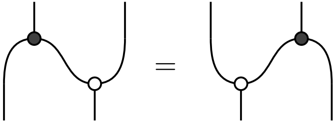

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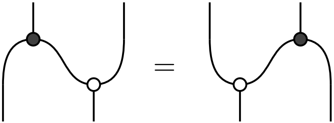

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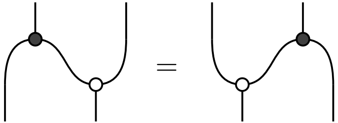

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  Define group algebra $\mu\colon g \otimes h \mapsto gh$, and $\eta\colon z \mapsto z \cdot 1_G$.

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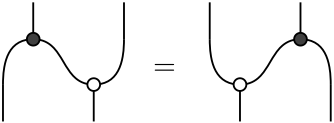

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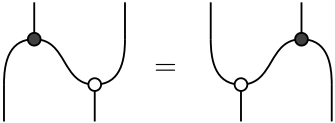

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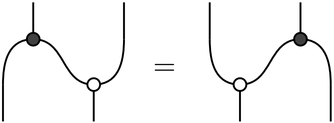

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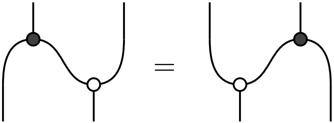

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Pair of pants

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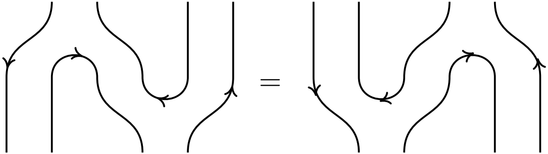

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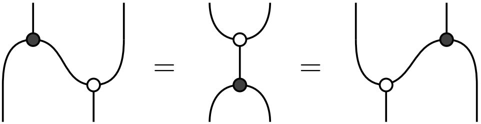
Any Frobenius structure satisfies:



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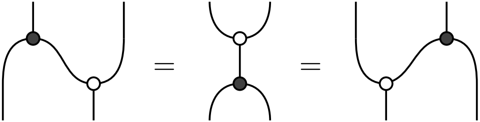

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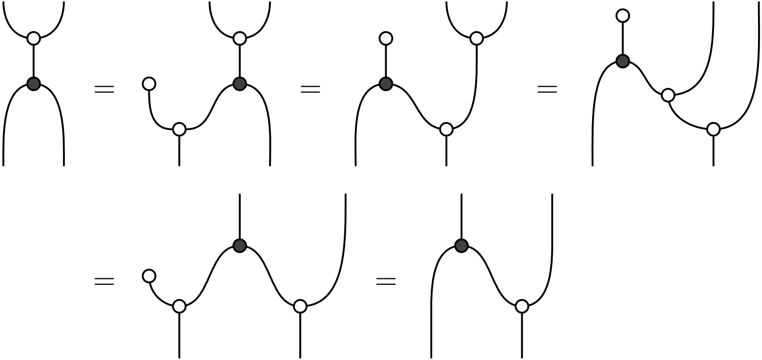
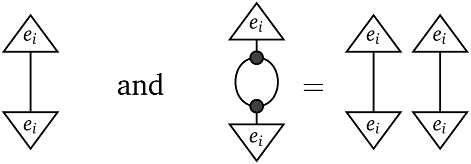


Speciality

If copies orthogonal basis $\{e_i\}$, can find (squared) norm of e_i:



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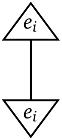

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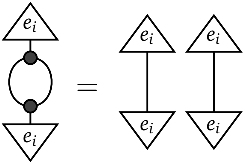

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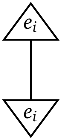

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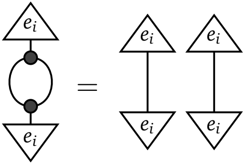

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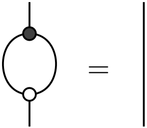

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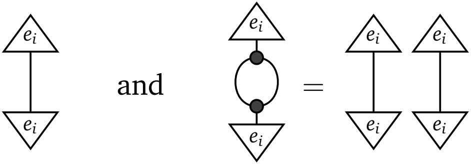

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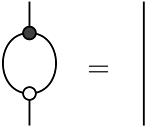

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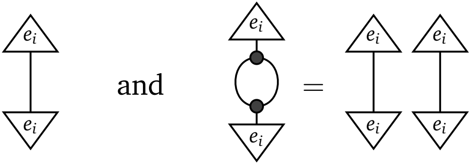

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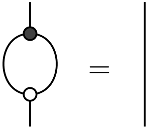

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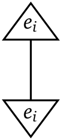

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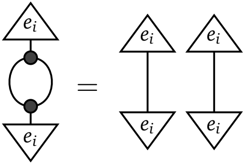

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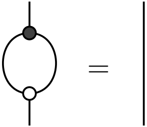

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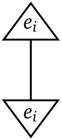

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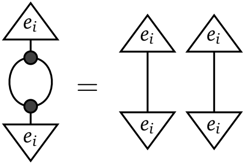

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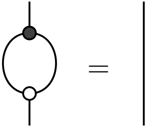

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- Groupoid Frobenius structure in **Rel** is always special

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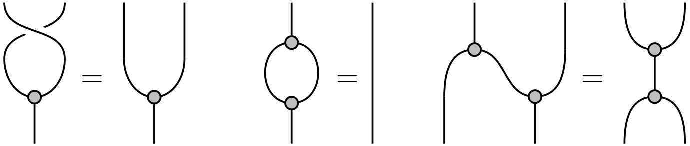
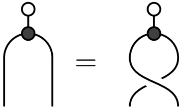
To check that $(A, \ , \ )$ is classical structure, only need:



Symmetry

Pair of pants hardly ever commutative. However:
A Frobenius structure is symmetric when:



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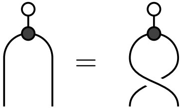

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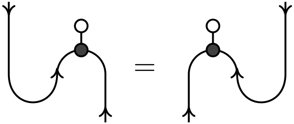

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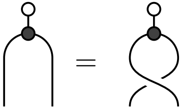

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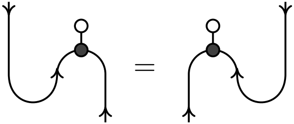

Examples:

- Pair of pants: in **FHilb** this says $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$
- Group algebras: inverses in groups are two-sided inverses
- Groupoid Frobenius structure: inverses are two-sided

Self-duality

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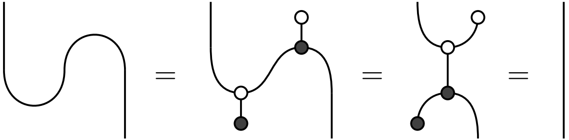
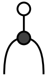


Nondegenerate forms

Monoid $(A, \,, \,)$ forms Frobenius structure with comonoid $(A, \,, \,)$ iff allows nondegenerate form: map $\colon A \to I$ with



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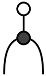

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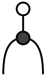

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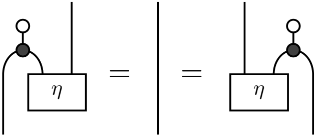

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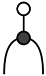

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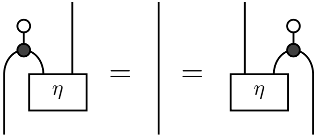
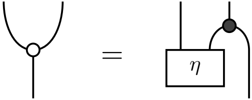


Define comultiplication as:



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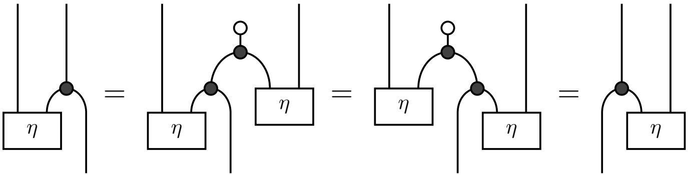

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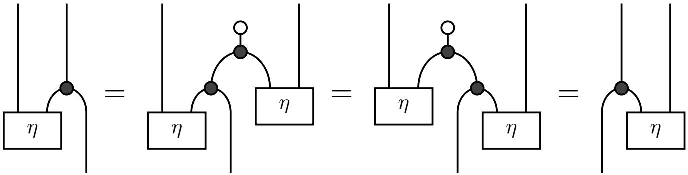
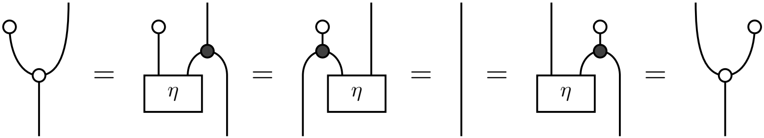


Counitality:



Nondegenerate forms

Proof (continued.)

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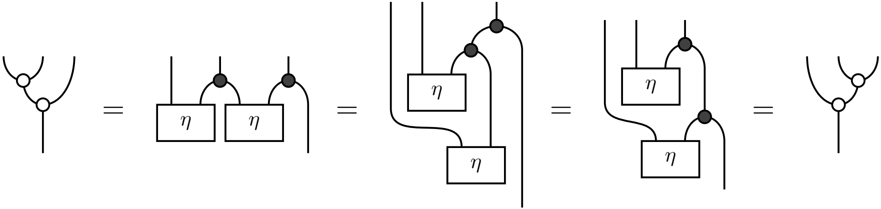

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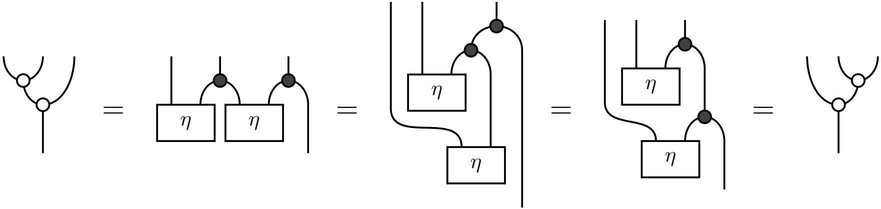
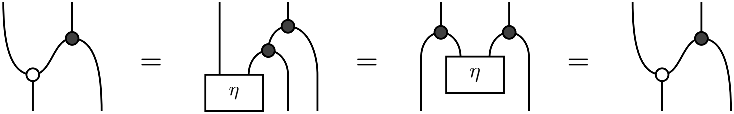


Frobenius law:



Homomorphisms

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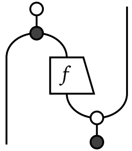
$f^{-1} =$ 

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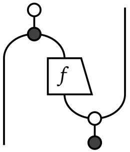
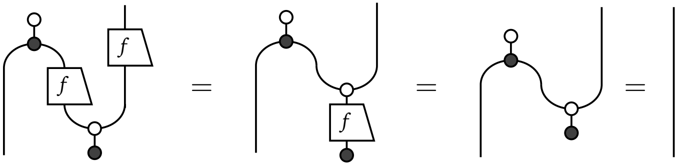
$f^{-1} =$ 

Indeed:



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Two ways to think about graphical calculus:

- diagram represents morphism:
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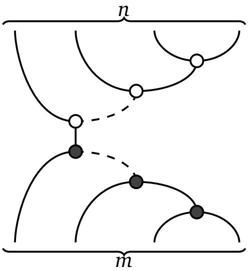
Unique way to copy (), discard (), fuse (), create () data!

Spider theorem

Let $(A, \ldots)$ be a special Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces, and id, using $\circ$ and $\otimes$, equals:



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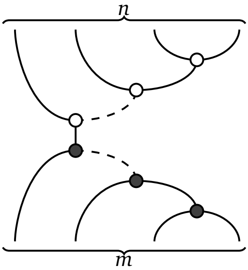

Proof. Induction on the number of dots.

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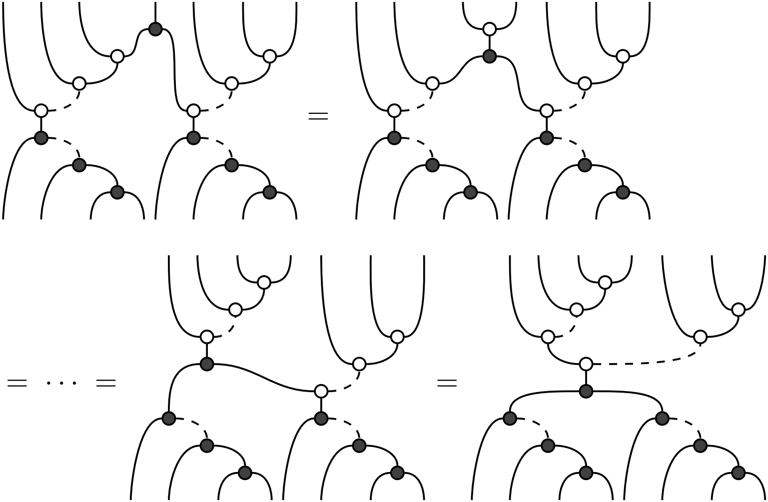
  Is the diagram underneath the connected?
  If so, use coassociativity and speciality.

Spider theorem

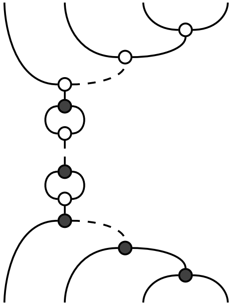
Proof. (continued.)

Suppose instead the rest of the diagram is disconnected:



More spider theorems

In a monoidal category, let $(A, \,,\,,\,,\,)$ be a Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , and id, using $\circ$ and $\otimes$, equals $(*)$.



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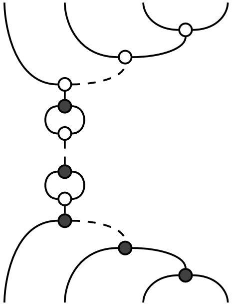

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In a symmetric monoidal category, let $(A, \;, \;, \;, \;)$ be a commutative Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , id, , using $\circ$ and $\otimes$, equals $(*)$.

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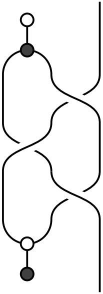

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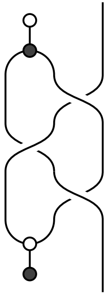

The Frobenius algebra axioms induce homotopy equivalences ('deformations') of the corresponding graph. Such moves cannot untie the knot.

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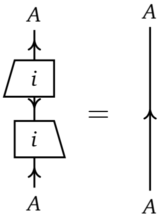

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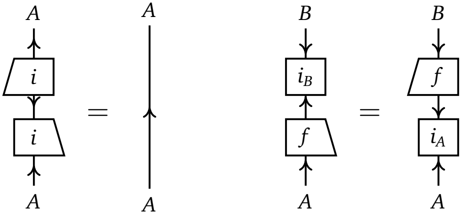

A morphism of involutive monoids is monoid homomorphism $A \xrightarrow{f} B$ satisfying $i_B \circ f = f_* \circ i_A$.

Example involutive monoids

- Matrix algebra. $\mathbb{M}_n$ is an involutive monoid in **FHilb**.
  Opposite monoid $\mathbb{M}_n^*$: multiplication ab in $\mathbb{M}_n^*$ is ba in $\mathbb{M}_n$.
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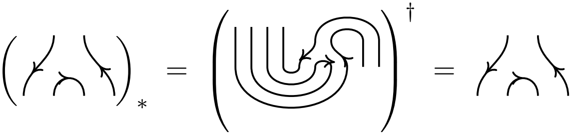

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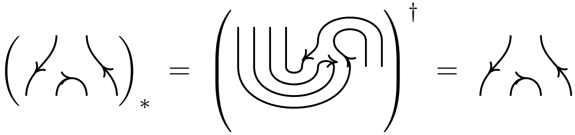
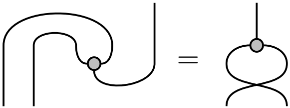


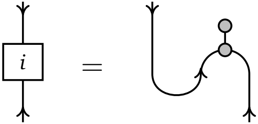
- Groupoids. **G** in **Rel** is involutive.
  Opposite monoid: induced by opposite groupoid $\mathbf{G}^{\mathrm{op}}$



Canonical involution $G \to G^*$ given by $g \sim g^{-1}$.

Frobenius law from way of the dagger

Monoid $(A,$ , $)$ is dagger Frobenius if and only if i is involution:



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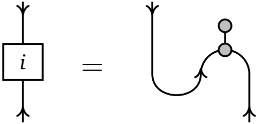

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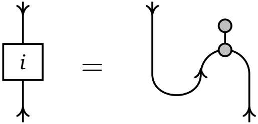

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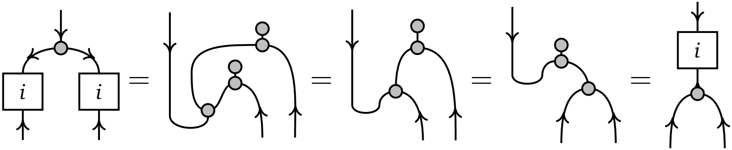

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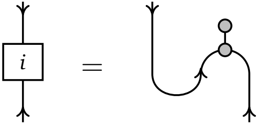

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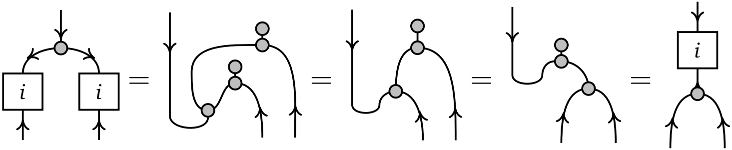

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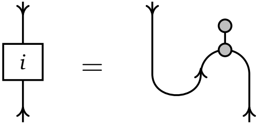

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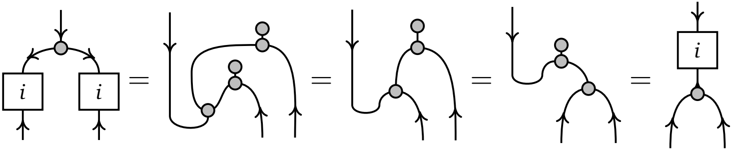

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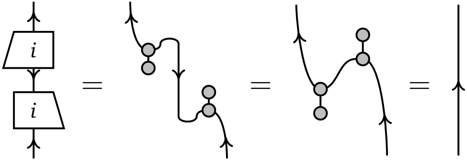

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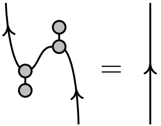

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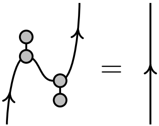

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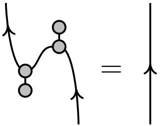
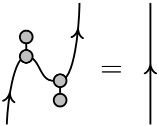
 and by applying $\dagger$, 

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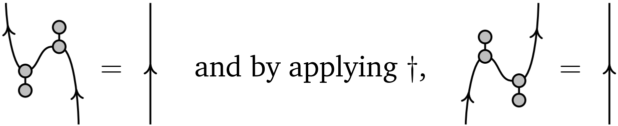

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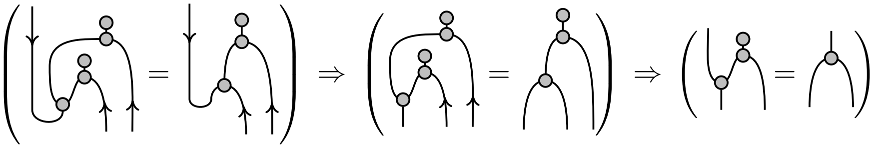

So the form definition gives rise to the correct comultiplication.

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Theorem: special dagger Frobenius structures in **FHilb** are of the form $\mathbb{M}_{n_1} \oplus \cdots \mathbb{M}_{n_k}$.

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Corollary: classical structure in **FHilb** copy orthonormal bases
Proof: must be of form $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

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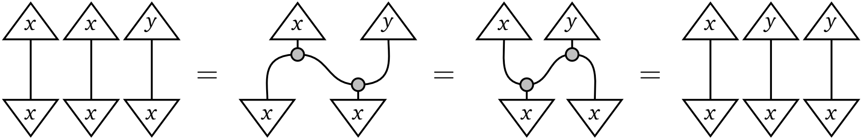

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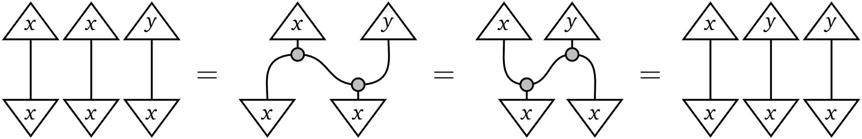

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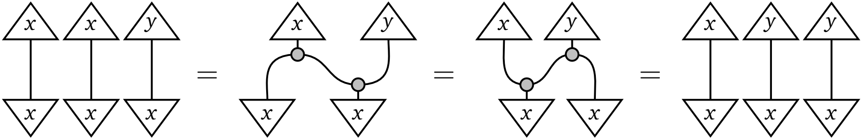

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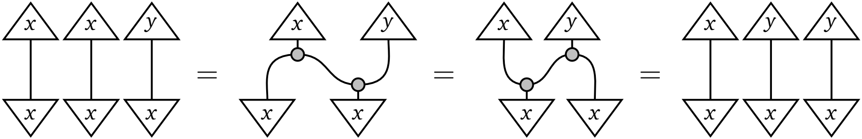

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Hence $\langle x-y|x-y\rangle = \langle x|x\rangle - \langle x|y\rangle - \langle y|x\rangle + \langle y|y\rangle = 0$, so $x = y$.

Orthogonal bases and morphisms

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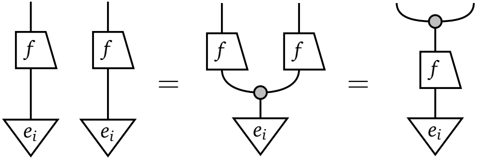
Proof. Suffices to see about basis of copyable states $\{e_i\}$.



Hence $f(e_i)$ copyable.

Classification in **Rel**

Theorem: Special dagger Frobenius structures in **Rel** correspond exactly to groupoids.

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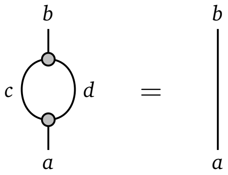
M is single-valued: by speciality $a(M \circ M^{\dagger})b$ iff $a = b$:



So: if $(c, d)Ma$ and $(c, d)Mb$, must have $a = b$.

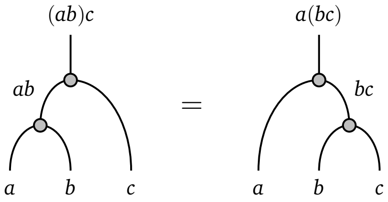
May simply write ab for unique c with $(a, b)Mc$.

Remember: ab not always defined!

Classification in **Rel**

Proof. (continued)

Associativity:

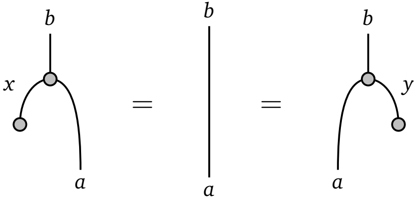


So ab and $(ab)c$ defined exactly when bc and $a(bc)$ are defined, and then $(ab)c = a(bc)$.

Classification in **Rel**

Proof. (continued)

Unitality: for units $x, y \in U$



So: a, b allow $x \in U$ with $xa = b$ iff $a = b$.
And: a, b allow $y \in U$ with $ay = b$ iff $a = b$.
If $z \in U$ then $xz = x$ for some $x \in U$. But then $x = z$!
Units idempotent; multiplication of different ones undefined.

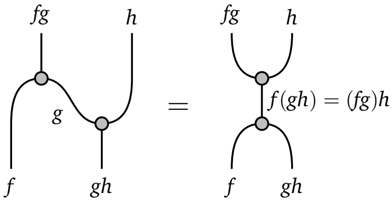
If $xa = a = x'a$, then $a = xa = x(x'a) = (xx')a$, so $x = x'$.
So every element has unique left/right identity.

Classification in **Rel**

Proof. (continued)

Category: U set of objects, A set of morphisms.

If fg defined and gh defined, want $(fg)h = f(gh)$ defined too:

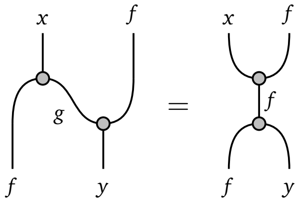


If fg and gh defined then LHS defined, so RHS defined too.

Classification in **Rel**

Proof. (continued)

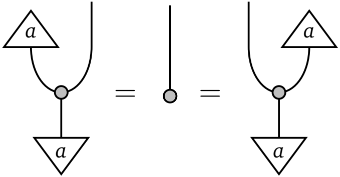
Inverses: for $f \in A$ with left unit x and right unit y:



□

Phases

Let $(A, \text{⅄}, \circ)$ be Frobenius structure in a monoidal dagger category.
State $I \xrightarrow{a} A$ is called phase when:



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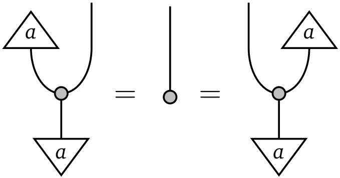
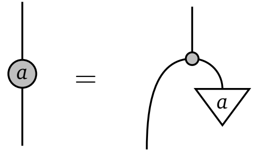


Its (right) phase shift is the following morphism $A \to A$:



Example phases

- For classical structure in **FHilb** copying basis $\{e_i\}$, vector $a = a_1 e_1 + \cdots a_n e_n$ is phase iff each a_i on unit circle: $|a_i|^2 = 1$.

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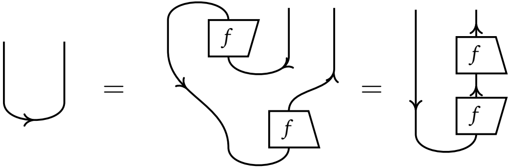
  Proof. The name of an morphism $A \xrightarrow{f} A$ is a phase when:

  But this means $f \circ f^\dagger = \mathrm{id}_A$; similarly $f^\dagger \circ f = \mathrm{id}_A$.

Example phases

- Phases of Frobenius structure $\mathbb{M}_n$ in **FHilb** form set $U(n)$ of n-by-n unitary matrices. Hence phases of $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ range over $U(k_1) \times \cdots \times U(k_n)$.

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- The phases of a Frobenius structure in **Rel** induced by a group G are elements of that group G itself.

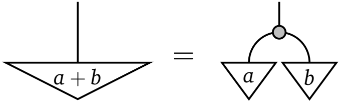
  Proof. For a subset $a \subseteq G$, equation defining phases reads

  $$\{g^{-1}h \mid g, h \in a\} = \{1_G\} = \{hg^{-1} \mid g, h \in a\}.$$

  So if $g \in G$, then $a = \{g\}$ is a phase. But if a contains distinct elements $g \neq h$ of G, cannot be phase. Similarly, $a = \emptyset$ not phase. Hence a phase precisely when singleton $\{g\}$.

Phase group

In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit  and:



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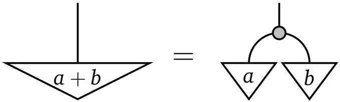
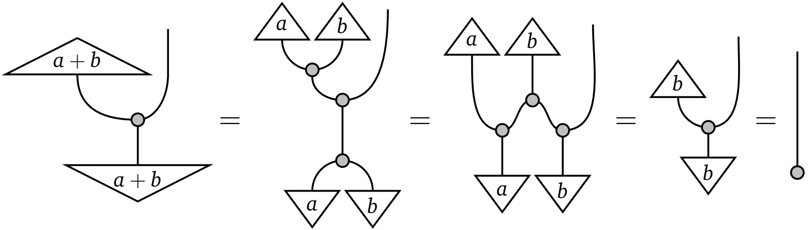


Proof. This is again a well-defined phase:



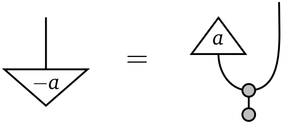
The flipped equation follows similarly.

Associativity is clear, hence phases form a monoid.

Phase group

Proof. (continued)

Left-inverse of phase a is:



Phase group

Proof. (continued)

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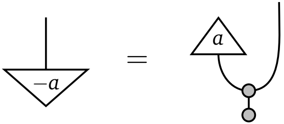
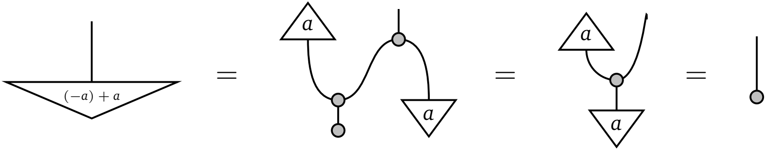


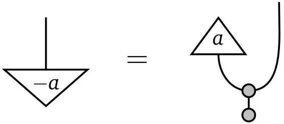
Left-inverse of a is $-a$:



Phase group

Proof. (continued)

Left-inverse of phase a is:



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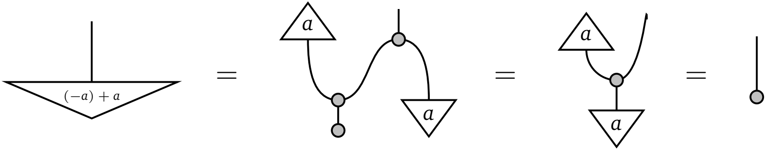

Similarly there is right-inverse. But in monoids, left and right inverses are equal: $l = l(xr) = (lx)r = r$. □

Example phase groups

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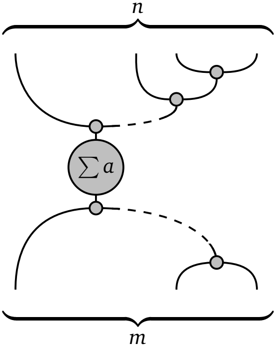
  In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.
- In **Rel**, the phase group induced by a group G is the group itself.

Phased spider theorem

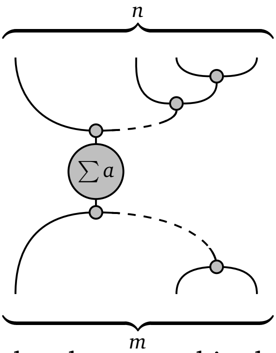
Let $(A, \cdot, \cdot)$ be classical structure in braided monoidal dagger category. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built of finitely many $\cdot, \cdot, \mathrm{id}, \sigma$ and phases using $\circ$, $\otimes$, and $\dagger$, equals



where a ranges over all the phases used in the diagram.

Phased spider theorem

Let $(A, \curlywedge, \multimap)$ be classical structure in braided monoidal dagger category. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built of finitely many $\curlywedge$, $\multimap$, id, σ and phases using $\circ$, $\otimes$, and $\dagger$, equals



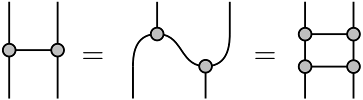
where a ranges over all the phases used in the diagram.

Proof. Use braidings to have all phases dangle at the bottom. Apply Spider Theorem. Use phase addition to reduce to single phase $\sum a$ on bottom right. Apply Spider Theorem again. $\square$

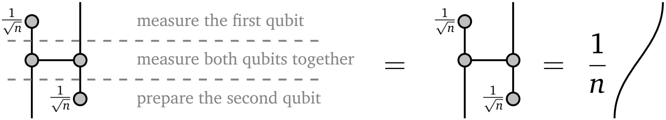
State transfer

State transfer protocol: transfer state of Hilbert space H from one system to another, with success probability $1/\dim(H)^2$.

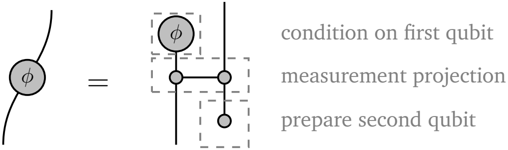
May be lax in drawing, e.g. projection $H \otimes H \to H \otimes H$:



The procedure looks like this:



Extra challenge: apply phase gate while transferring state



Summary

- Frobenius structures: interacting co/monoid, self-duality
- Normal forms: spider theorems
- Frobenius law: justified by coherence
- Classification: matrix algebras, bases, groupoids
- Phases: unitary operators, state transfer

Next week: interaction between two Frobenius structures