Categories and Quantum Informatics exercise sheet 6: Frobenius structures

Exercise 5.1. Recall that in a braided monoidal category, the tensor product of monoids is again a monoid.

(a) Show that, in a braided monoidal category, the tensor product of Frobenius structures is again a Frobenius structure.

(b) Show that, in a symmetric monoidal category, the tensor product of symmetric Frobenius structures is again a symmetric Frobenius structure.

(c) Show that, in a symmetric monoidal dagger category, the tensor product of classical structures is again a classical structure.

Exercise 5.2. This exercise is about the interdependencies of the defining properties of Frobenius structures in a braided monoidal dagger categories. Recall the Frobenius law (5.1).

(a) Show that for any maps $A \xrightarrow{d} A \otimes A$ and $A \otimes A \xrightarrow{m} A$, speciality ($m \circ d = \mathrm{id}$) and equation (5.4) together imply associativity for m.

(b) Suppose $A \xrightarrow{d} A \otimes A$ and $A \otimes A \xrightarrow{m} A$ satisfy equation (5.4), speciality, and commutativity (4.7). Given a dual object $A \dashv A^*$, construct a map $I \xrightarrow{u} A$ such that unitality (4.6) holds.

Exercise 5.3. Recall that a set $\{x_0, \ldots, x_n\}$ of vectors in a vector space is *linearly independent* when $\sum_{i=0}^{n} z_i x_i = 0$ for $z_i \in \mathbb{C}$ implies $z_0 = \ldots = z_n = 0$. Show that the nonzero copyable states of a comonoid in **FHilb** are linearly independent. (Hint: consider a minimal linearly dependent set.)

Exercise 5.4. This exercise is about the phase group of a Frobenius structure in **Rel** induced by a groupoid.

(a) Show that a phase of $\mathbf{G}$ corresponds to a subset of the arrows of $\mathbf{G}$ that contains exactly one arrow out of each object and exactly one arrow into each object.

(b) A *cycle* in a category is a series of morphisms $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \cdots A_n \xrightarrow{f_n} A_1$. For finite $\mathbf{G}$, show that a phase corresponds to a union of cycles that cover all objects of $\mathbf{G}$. Find a phase on the indiscrete category on $\mathbb{Z}$ that is not a union of cycles.

(c) A groupoid is *totally disconnected* when all morphisms are endomorphisms. Show that for such groupoids $\mathbf{G}$, the phase group is $\mathbf{G}$ itself, regarded as a group: $\prod_{x \in \mathrm{Ob}(\mathbf{G})} \mathbf{G}(x, x)$. Conclude that this holds in particular for classical structures.