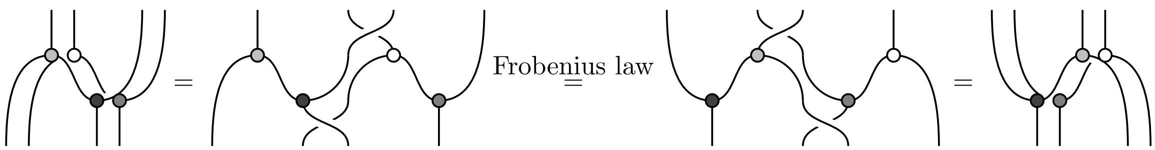
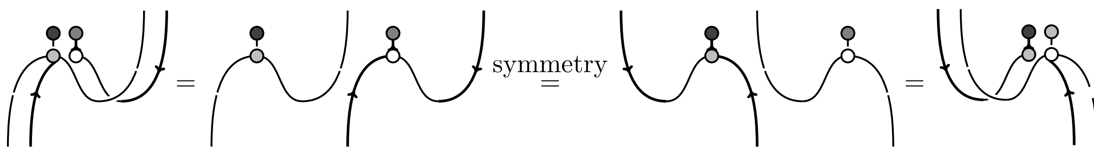
Categories and Quantum Informatics exercise sheet 6: Frobenius structures

Exercise 5.1. (a) We have seen before that the tensor product of a monoid is again a monoid. The same holds for comonoids. It is left to verify the Frobenius law:



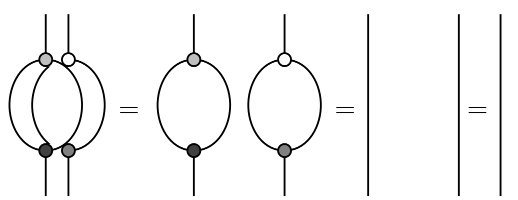
(1)

(b) We will use the fact that the tensor product $A \otimes B$ of two spaces A, B that have duals A^*, B^*, is dual to the tensor product $A^* \otimes B^*$. We use the alternative definition of symmetric Frobenius algebras in symmetric monoidal categories; however, it can also be shown directly.



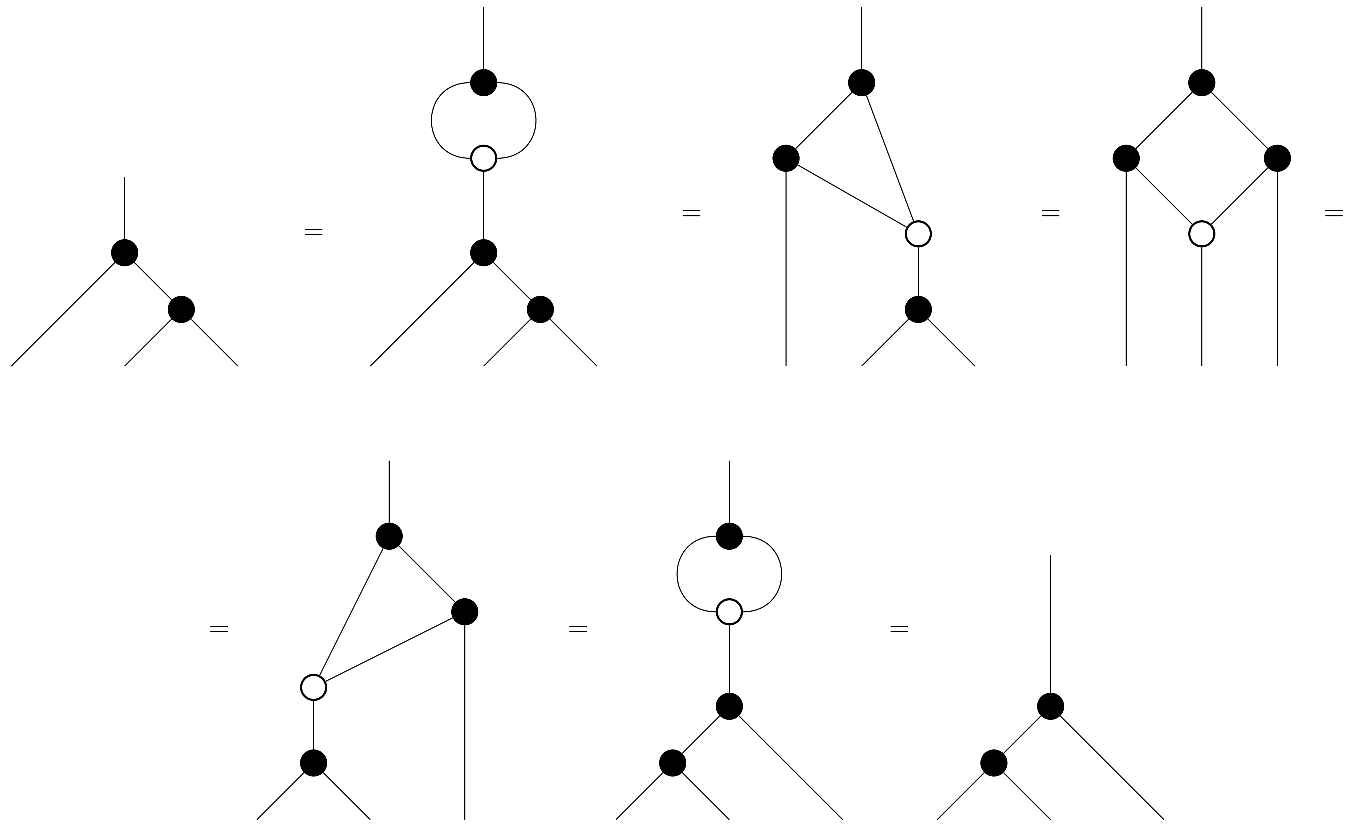
(2)

(c) The tensor product of commutative frobenius structures is again a commutative frobenius structure by (a) and an argument similar to (b). It is left to show that the tensor product of special dagger Frobenius algebras is special.

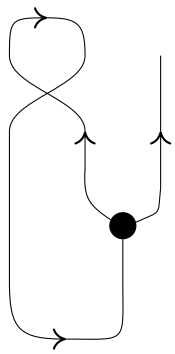


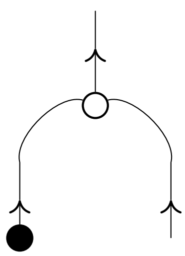
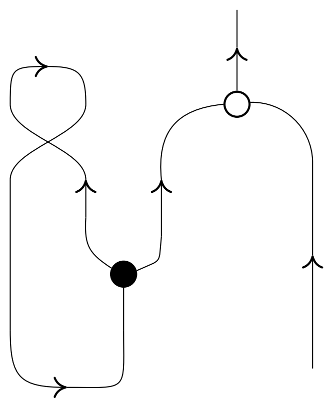
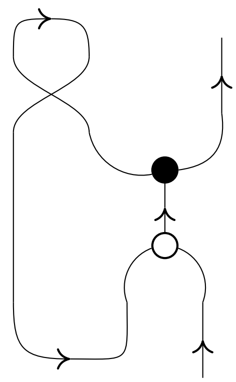
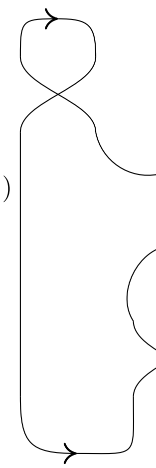
(3)

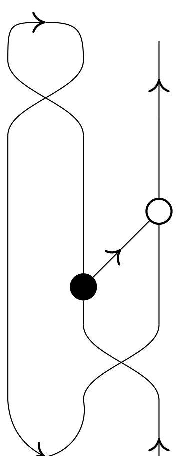
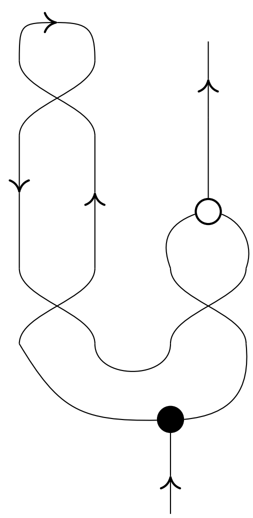
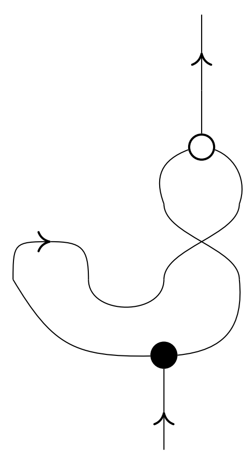
Exercise 5.2. (a)

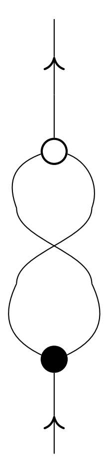
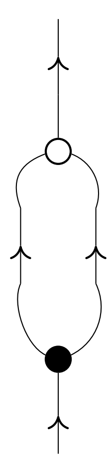


(b)

Define  :=  Then:

 =  (Frob.) =  (comm.) = 

(Frob.) =  (naturality) =  (swap invert.) = 

(snake) =  (comm) =  (speciality) = 

Exercise 5.3. Suppose $\{x_0, \dots, x_n\}$ is a minimal nonempty linearly dependent set of nonzero copyable states. Then $x_0 = \sum_{i=1}^{n} z_i x_i$ for suitable coefficients $z_i \in \mathbb{C}$. So

$$\begin{aligned}\sum_{i=1}^{n} z_i(x_i \otimes x_i) &= \sum_{i=1}^{n} z_i d(x_i) \\ &= d(x_0) \\ &= (\sum_{i=1}^{n} z_i x_i) \otimes (\sum_{j=1}^{n} z_j x_j) \\ &= \sum_{i,j=1}^{n} z_i z_j (x_i \otimes x_j).\end{aligned}$$

By minimality, $\{x_1, \dots, x_n\}$ is linearly independent. Hence $z_i^2 = z_i$ for all i, and $z_i z_j = 0$ for $i \neq j$. So $z_i = 0$ or $z_i = 1$ for all i. If $z_j = 1$, then $z_i = 0$ for all $i \neq j$, so $x_0 = x_j$. By minimality, then $j = 1$ and $\{x_0, x_j\} = \{x_0\}$, which is impossible. So we must have $z_i = 0$ for all i. But then $x_0 = 0$, which is likewise a contradiction.

Exercise 5.4. (a) The defining equation for phases gives

$$\{g^{-1} \circ h \mid g, h \in a\} = \{\mathrm{id}_x \mid x \in \mathrm{Ob}(\mathbf{G})\}\{g \circ h^{-1} \mid g, h \in a\}.$$

The inclusion $L \subseteq M$ means: $\forall g, h \in a\colon\ \mathrm{cod}(g) = \mathrm{cod}(h) \implies g = h$. The inclusion $M \supseteq R$ means: $\forall g, h \in a\colon\ \mathrm{dom}(g) = \mathrm{dom}(h) \implies g = h$. In other words: there can be at most one arrow in a out of each object of $\mathbf{G}$, and at most one arrow of a into each object of $\mathbf{G}$. Given this, the remaining inclusions $L \supseteq M \subseteq R$ mean: $\forall x \in \mathrm{Ob}(\mathbf{G}) \exists g, h \in a\colon\ \mathrm{dom}(g) = x = \mathrm{cod}(h)$. That is: a contains arrows into and out of each object.

(b) Pick an object x; the phase a contains exactly one arrow $x \longrightarrow y$. If $y = x$, we have a 1-cycle. Otherwise, a contains exactly one arrow $y \longrightarrow z$, etc. This process has to end, because the groupoid is finite. Delete all the objects involved in the cycle, and repeat.
For the indiscrete groupoid on $\mathbb{Z}$, there is a phase $\{n \xrightarrow{!} n+1 \mid n \in \mathbb{Z}\}$