Categories and Quantum Informatics

Week 5: Monoids and comonoids

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informatics

Overview

- Monoids: multiplication of states
- Comonoids: ‘copying’ of states
- Cloning: prove no-cloning and no-deleting
- Products: characterize when tensor product is product

Copying

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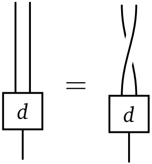

cocommutativity

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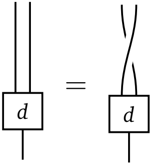

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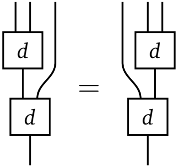

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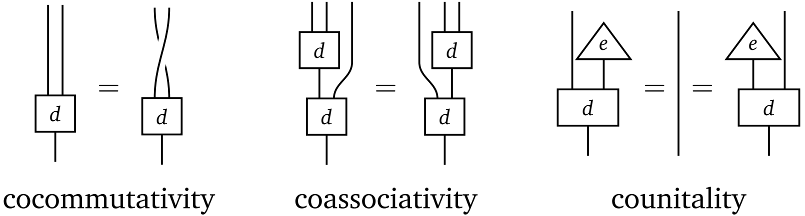

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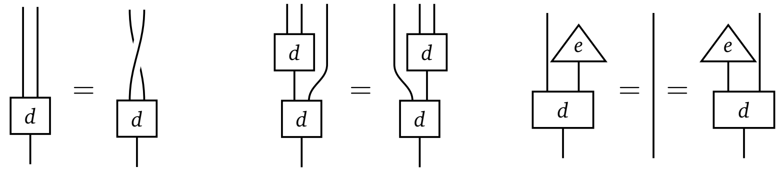

cocommutativity coassociativity counitality

Triple (A, d, e) is called (cocommutative) comonoid.

Example comonoids

- In **Set**, the tensor product is a Cartesian product.
  Every object carries a unique comonoid with comultiplication $a \mapsto (a, a)$ and counit $a \mapsto \bullet$, which is cocommutative.

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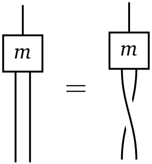
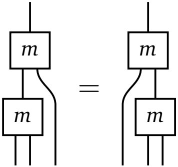
- In **FHilb**, basis $\{e_i\}$ for a Hilbert space gives a cocommutative comonoid, with comultiplication $e_i \mapsto e_i \otimes e_i$ and counit $e_i \mapsto 1$.

Monoids

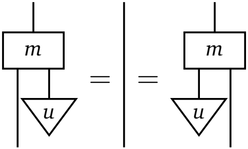
Dually:



commutativity



associativity



unitality

Monoids

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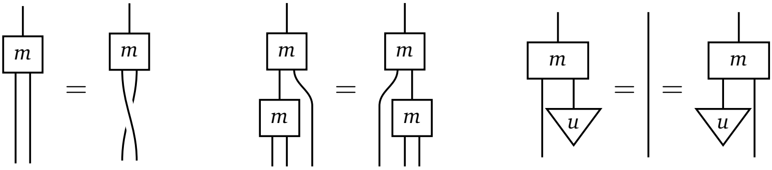

commutativity *associativity* *unitality*

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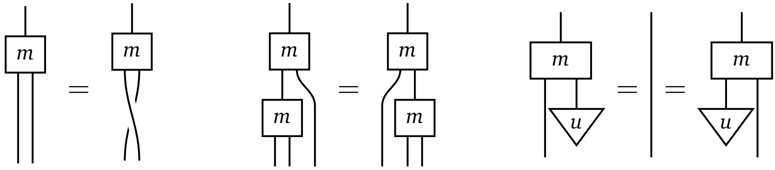

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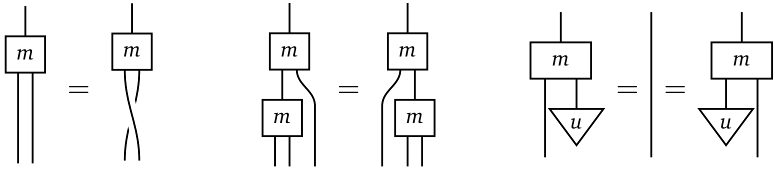

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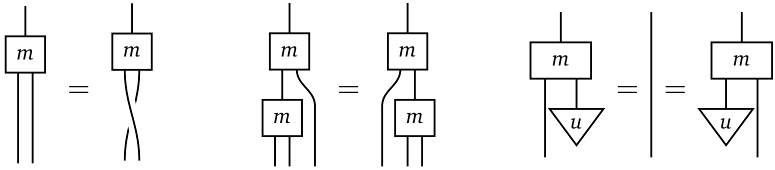

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- Tensor unit I, with multiplication $\rho_I = \lambda_I$ and unit id_I.
- A monoid in **Set** is just an ordinary monoid; e.g. any group.
- A monoid in **Vect** is an *algebra*: a set where we can add vectors and multiply with scalars, and also multiply vectors bilinearly. E.g. $\mathbb{C}^n$ under pointwise multiplication and unit $(1, 1, \ldots, 1)$. E.g. vector space of n-by-n matrices with matrix multiplication.

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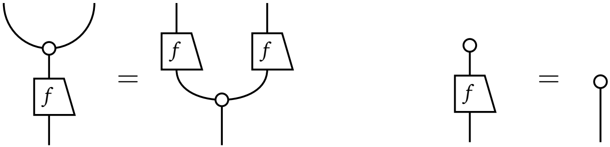

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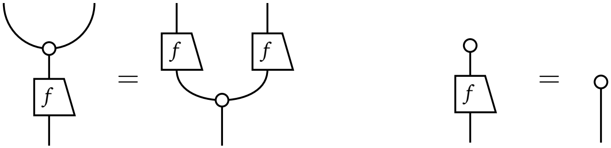

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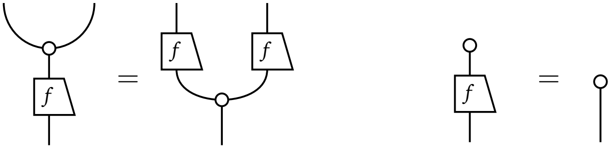

Dually: monoid homomorphism.

Given monoidal category, can build new category of (co)monoids and homomorphisms.

Example homomorphisms

- In **Set**, any function $A \xrightarrow{f} B$ is a comonoid homomorphism: $(f \times f)(a, a) = \big(f(a), f(a)\big)$, and $f(a) = \bullet$.

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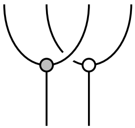
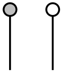
- In **FHilb**, any function $\{d_i\} \xrightarrow{f} \{e_j\}$ between bases extends linearly to a comonoid homomorphism: $d(f(d_i)) = f(d_i) \otimes f(d_i)$ and $e(f(d_j)) = 1 = e(d_j)$.

Product of monoids

Can combine two (co)monoids to single one using braiding:





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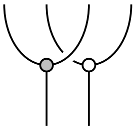

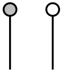

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- In **FHilb**, the product of comonoids on H and K that copy bases $\{d_i\}$ and $\{e_j\}$ is the comonoid copying basis $\{d_i \otimes e_j\}$ of $H \otimes K$.

Dagger

Monoidal dagger category has duality between monoids and comonoids: (A, d, e) is a comonoid if and only if $(A, d^\dagger, e^\dagger)$ is a monoid.

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Closure

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Can handle this using names and conames. E.g.:

$$\mathbf{FHilb}(H, K) = \{H \xrightarrow{f} K \mid f \text{ linear}\}$$

is vector space with pointwise operations $(f+g)(x) = f(x) + g(x)$,
Hilbert space with *trace inner product* $\langle f|g\rangle = \mathrm{Tr}(f^\dagger \circ g)$.

To transform morphisms, encode them as vectors in function spaces.

Matrices

One of most important features of matrices: they can be multiplied.
In other words, linear maps $\mathbb{C}^n \to \mathbb{C}^n$ can be composed.
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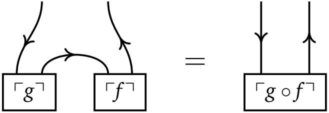
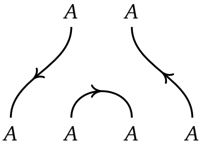
More generally, if an object A in a monoidal category has a dual A^*, then operators $A \xrightarrow{f} A$ correspond bijectively to states $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes A$.
Composition $A \xrightarrow{g \circ f} A$ of operators transfers to states $I \xrightarrow{\ulcorner g \circ f \urcorner} A^* \otimes A$:

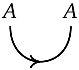


So $A^* \otimes A$ canonically becomes monoid.

Pair of pants

If $A \dashv A^*$ in monoidal category, then $A^* \otimes A$ is a monoid:





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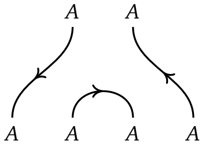
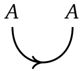
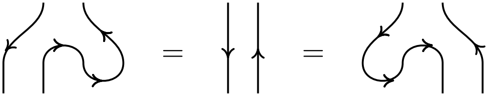
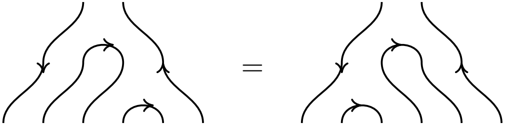




Proof.





$\square$

Matrix algebras

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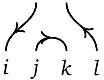
This bijection respects multiplication:



$$= \begin{bmatrix} \langle i| \otimes |l\rangle & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix} \longmapsto \begin{bmatrix} e_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix} = e_{ij}e_{kl}$$

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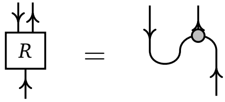
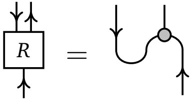
Abstract embedding of $(M, \;, \;)$ into $M \dashv M^*$:



Cayley's theorem

Any monoid $(A, \ , \ )$ in a monoidal category with $A \dashv A^*$ has monoid homomorphism to $(A^* \otimes A, \ , \ )$ with right inverse.



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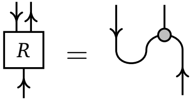

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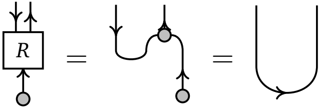

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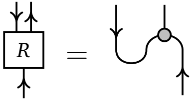

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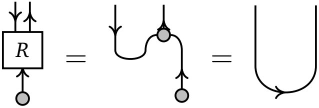
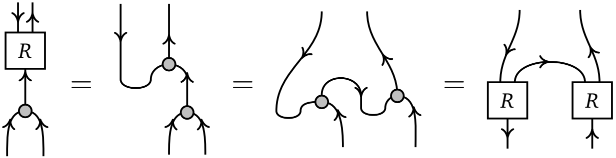


R preserves multiplication:



Finally, *R* has a right inverse . □

Uniform deleting

Counit $A \xrightarrow{e} I$ tells us we can 'delete' A if we want to.
What does it mean to have deletion *systematically* on every object?

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A monoidal category has uniform deleting if there is a natural transformation $A \xrightarrow{e_A} I$ with $e_I = \mathrm{id}_I$, such that:

$$
\begin{tikzcd}
 & A \otimes B \arrow[dl, "e_A \otimes e_B"'] \arrow[dr, "e_{A \otimes B}"] & \\
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Proof. Uniform deleting gives a morphism $A \xrightarrow{e_A} I$ for each object A.
Naturality and $e_I = \mathrm{id}_I$ then show any morphism $A \xrightarrow{f} I$ equals e_A.
Conversely, if I is terminal, choose $e_A : A \to I$ uniquely. $\square$

No-deleting theorem

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Theorem: if a monoidal category with duals has uniform deleting, then it is a preorder.

Proof. Let $A \xrightarrow{f,g} B$ be morphisms. Naturality of e gives:

$$\begin{tikzcd}
A \otimes B^* \arrow[r, "e_{A\otimes B^*}"] \arrow[d, "\llcorner f \lrcorner"'] & I \arrow[d, "\mathrm{id}_I"] \\
I \arrow[r, "e_I = \mathrm{id}_I"] & I
\end{tikzcd}$$

So $\llcorner f \lrcorner = e_{A\otimes B^*}$, and similarly $\llcorner g \lrcorner = e_{A\otimes B^*}$. Hence $f = g$. $\square$

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Answer: copying must respect composition, tensor products.

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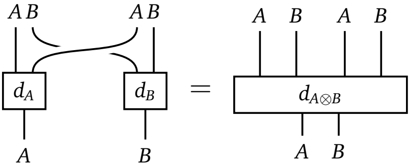
A braided monoidal category has uniform copying if there is a natural transformation $A \xrightarrow{d_A} A \otimes A$ with $d_I = \rho_I$, satisfying cocommutativity and coassociativity, and:



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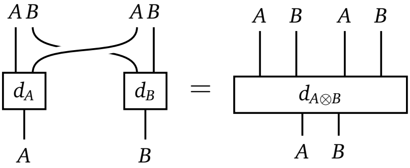
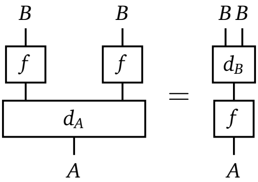
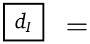


Naturality and $d_I = \rho_I$ look like this for arbitrary $A \xrightarrow{f} B$:





Copying states

Example: **Set** has uniform copying maps $a \mapsto (a, a)$:

$d_1(\bullet) = (\bullet, \bullet) = \rho_1(\bullet)$

both maps $A \times B \to A \times B \times A \times B$ are $(a, b) \mapsto (a, b, a, b)$

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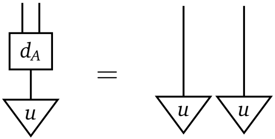
In a braided monoidal category, a state $I \xrightarrow{u} A$ is copyable with respect to a map $A \xrightarrow{d_A} A \otimes A$ when:



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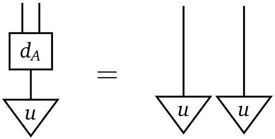

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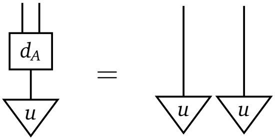
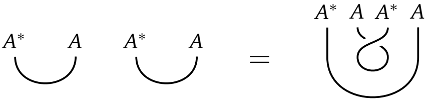


In braided monoidal category with uniform copying, any state is copyable.

Proof. If there is uniform copying, then, by naturality of the copying maps, we have $d_A \circ u = (u \otimes u) \circ \rho_I$ for each state $I \xrightarrow{u} A$. $\square$

Duals vs copying

If a braided monoidal category with duals has uniform copying:



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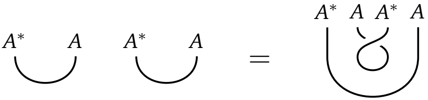

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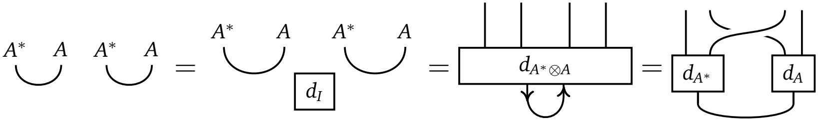

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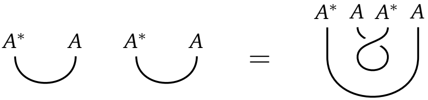

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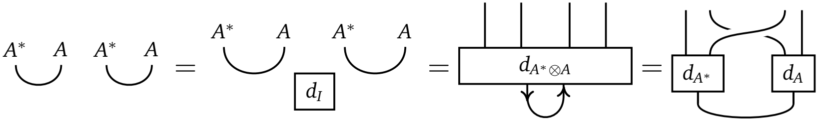

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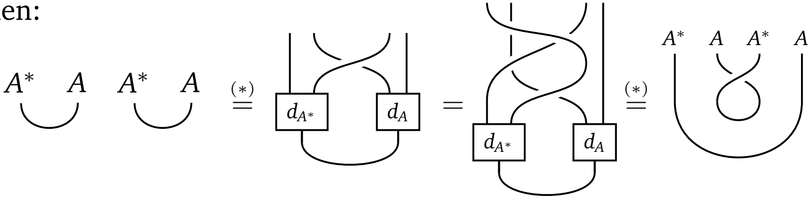
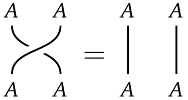


Duals vs copying

In a braided monoidal category with duals and uniform copying:



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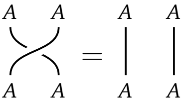
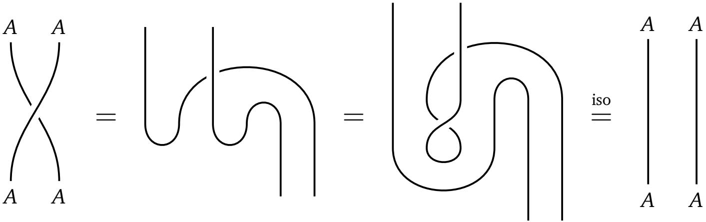
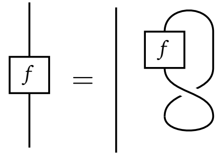


Proof.



No-cloning theorem

If a braided monoidal category with duals has uniform copying, every endomorphism is a multiple of the identity, $f = \mathrm{Tr}(f) \bullet \mathrm{id}$:



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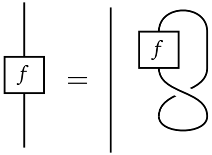

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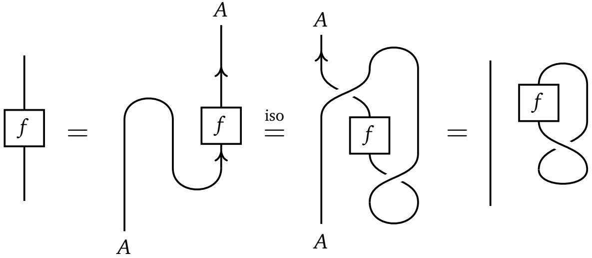

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- tensor products are products and the tensor unit is terminal
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For converse, need to prove $A \otimes B$ is product of A, B.
For $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$, define

$$\begin{pmatrix} f \\ g \end{pmatrix} = (f \otimes g) \circ d$$

$$p_A = \rho_A \circ (\mathrm{id}_A \otimes e_B) \colon A \otimes B \longrightarrow A$$

$$p_B = \lambda_B \circ (e_A \otimes \mathrm{id}_B) \colon A \otimes B \longrightarrow B$$

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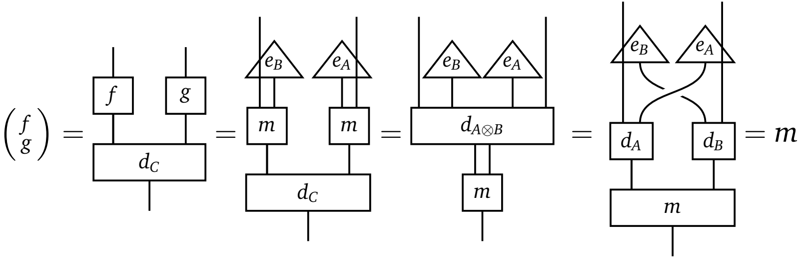

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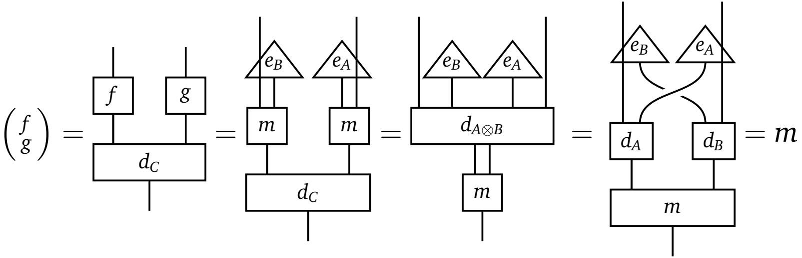

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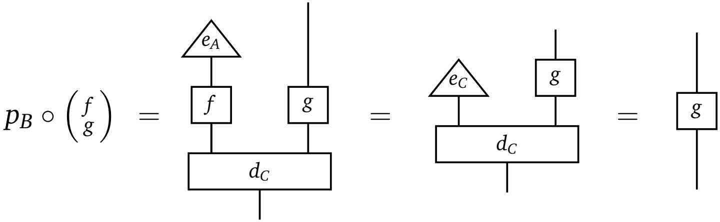
Finally, we show the universal morphism has the right properties:



A similar result holds for g. $\square$

Summary

- Monoids: multiplication on states
- Comonoids: ‘copying’ of states
- Closure: operators form monoids
- Cloning: no-cloning and no-deleting
- Products: characterize when tensor product is product

Next week: interaction between monoids and comonoids