Categories and Quantum Informatics
Week 5: Monoids and comonoids

Chris Heunen
Overview

- Monoids: multiplication of states
- Comonoids: ‘copying’ of states
- Cloning: prove no-cloning and no-deleting
- Products: characterize when tensor product is product
Copying

What does copying object $A$ mean?
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- shouldn’t matter if we switch both output copies

\[
\text{cocommutativity}
\]
Copy

What does **copying** object $A$ mean? Type should be $A \xrightarrow{d} A \otimes A$

- shouldn’t matter if we switch both output copies
- if copying twice, shouldn’t matter if take first or second copy

\[
d \quad = \quad d
\]

cocommutativity

\[
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\]

coassociativity
What does copying object $A$ mean? Type should be $A \xrightarrow{d} A \otimes A$

- shouldn’t matter if we switch both output copies
- if copying twice, shouldn’t matter if take first or second copy
- output should equal input: uses deletion $A \xrightarrow{e} I$

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{d} \\
\end{array}
\end{array}
&= \\
\begin{array}{c}
\begin{array}{c}
\xrightarrow{d} \\
\end{array}
\end{array}
\end{align*}
\]

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\end{array}
\end{array}
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\[
\begin{align*}
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\xrightarrow{e} \\
\end{array}
\end{array}
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\begin{array}{c}
\begin{array}{c}
\xrightarrow{e} \\
\end{array}
\end{array}
\end{align*}
\]

cocommutativity \quad \text{coassociativity} \quad \text{counitinality}
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\[
\begin{align*}
\begin{array}{ccc}
\xrightarrow{d} & = & \xrightarrow{d} \\
\xrightarrow{d} & = & \xrightarrow{d} \\
\end{array}
\end{align*}
\]

- cocommutativity
- coassociativity
- counitality

Triple $(A, d, e)$ is called (**cocommutative**) comonoid.
Example comonoids

- In $\textbf{Set}$, the tensor product is a Cartesian product. Every object carries a unique comonoid with comultiplication $a \mapsto (a, a)$ and counit $a \mapsto \bullet$, which is cocommutative.
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- In **Rel**, any group $G$ forms a comonoid with comultiplication $g \sim (h, h^{-1}g)$ and counit $1 \sim \bullet$. 
  
  **Counitality:** LHS is $g \sim h$ where $h^{-1}g = 1$, RHS is $g \sim 1^{-1}g$. The comonoid is cocommutative iff the group is abelian. 
  
  **Cocommutativity:** LHS is $g \sim (h^{-1}g, h)$, RHS is $g \sim (k, k^{-1}g)$. 

- In **FHilb**, basis $\{e_i\}$ for a Hilbert space gives a cocommutative comonoid, with comultiplication $e_i \mapsto e_i \otimes e_i$ and counit $e_i \mapsto 1$. 


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Monoids

Dually:

\[
\begin{align*}
    & m = m \\
    & m = m \\
    & m = m \\
\end{align*}
\]

- commutativity
- associativity
- unitality

Examples:

- Tensor unit \( I \), with multiplication \( \rho I = \lambda I \) and unit \( I \).
- A monoid in \( \text{Set} \) is just an ordinary monoid; e.g. any group.
- A monoid in \( \text{Vect} \) is an algebra: a set where we can add vectors and multiply with scalars, and also multiply vectors bilinearly. E.g. \( \mathbb{C}^n \) under pointwise multiplication and unit \((1, 1, \ldots, 1)\).
- E.g. vector space of \( n \times n \) matrices with matrix multiplication.
Monoids

Dually:

\[
\begin{align*}
&m = m \\
&m = m \\
&m = m \\
&u = u
\end{align*}
\]

*commutativity* \hspace{1cm} *associativity* \hspace{1cm} *unitality*

Triple \((A, m, u)\) is (commutative) monoid.
Monoids

Dually:

\[
\begin{align*}
&\text{commutativity} & = & & \text{associativity} & = & & \text{unitality} \\
&m & = & m & = & m & = & m
\end{align*}
\]

Triple \((A, m, u)\) is (commutative) monoid. Examples:

- Tensor unit \(I\), with multiplication \(\rho_I = \lambda_I\) and unit \(\text{id}_I\).
Monoids

Dually:

\[
\begin{align*}
  m \ast m &= m \\
  m \ast m &= m \\
  \text{commutativity} & \quad \text{associativity} & \quad \text{unitality}
\end{align*}
\]

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m & = m \\
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\end{align*}
\]

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\text{commutativity} & \quad \text{associativity} \\
\quad & \quad \\
\end{align*}
\]

\[
\begin{align*}
m & = m \\
\text{unitality} & = \quad \\
\quad & = \\
\end{align*}
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Homomorphisms

Draw comultiplication as \( \mathcal{V} \), counit as \( \mathfrak{q} \), multiplication as \( \mathfrak{h} \), unit as \( \mathfrak{d} \).
Homomorphisms

Draw comultiplication as $\nabla$, counit as $\varepsilon$, multiplication as $\otimes$, unit as $\mathbb{1}$. Choosing bases $\{d_i\}$ and $\{e_j\}$ makes $H$ and $K$ in $\text{FHilb}$ comonoids.
Homomorphisms

Draw comultiplication as $\nabla$, counit as $\wp$, multiplication as $\otimes$, unit as $\mathbb{1}$.

Choosing bases $\{d_i\}$ and $\{e_j\}$ makes $H$ and $K$ in $\text{FHilb}$ comonoids.

Functions $\{d_i\} \rightarrow \{e_j\}$ respect comultiplication and counit.
Homomorphisms

Draw comultiplication as $\gamma'$, counit as $\varphi$, multiplication as $\otimes$, unit as $\mathbb{1}$.

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Functions $\{d_i\} \to \{e_j\}$ respect comultiplication and counit.

A comonoid homomorphism $(A, \gamma', \varphi) \to (B, \gamma', \varphi)$ is $A \xrightarrow{f} B$ with:

\[
\begin{align*}
\bullet & \quad = \\
\begin{array}{c}
\text{f} \\
\end{array} & \quad \begin{array}{c}
f \\
\end{array} & \quad \begin{array}{c}
f \\
\end{array}
\end{align*}
\]

Dually: monoid homomorphism.

Given monoidal category, can build new category of (co)monoids and homomorphisms.
Homomorphisms

Draw comultiplication as $\nabla$, counit as $\wp$, multiplication as $\odot$, unit as $\bullet$. Choosing bases $\{d_i\}$ and $\{e_j\}$ makes $H$ and $K$ in $\textbf{FHilb}$ comonoids.

Functions $\{d_i\} \to \{e_j\}$ respect comultiplication and counit.

A comonoid homomorphism $(A,\nabla,\wp) \to (B,\nabla,\wp)$ is $A \xrightarrow{f} B$ with:

\[
\begin{align*}
\begin{tikzpicture}
    \node (a) at (0,0) {$f$};
    \node (b) at (1,0) {$f$};
    \node (c) at (2,0) {$f$};
    \node (d) at (0,-1) {$\nabla$};
    \node (e) at (1,-1) {$\odot$};
    \node (f) at (2,-1) {$\wp$};
    \draw (a) -- (d);
    \draw (b) -- (e);
    \draw (c) -- (f);
\end{tikzpicture}
\end{align*}
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Given monoidal category, can build new category of (co)monoids and homomorphisms.
Example homomorphisms

- In **Set**, any function $A \xrightarrow{f} B$ is a comonoid homomorphism:
  $$(f \times f)(a, a) = (f(a), f(a)),$$ and $f(a) = \bullet$. 

- In **Rel**, any surjective homomorphism $G \xrightarrow{f} H$ of groups is a comonoid homomorphism. Preservation of comultiplication:
  LHS is $g \sim (h, h^{-1}f(g))$, RHS is $g \sim (f(g'), f(g')^{-1}f(g))$.

- In **FHilb**, any function $\{d_i\} \xrightarrow{f} \{e_j\}$ between bases extends linearly to a comonoid homomorphism:
  $d(f(d_i)) = f(d_i) \otimes f(d_i)$ and $e(f(d_j)) = 1 = e(d_j)$. 

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Product of monoids

Can combine two (co)monoids to single one using braiding:

- In \(\text{Set}\), product comonoid on \(A, B\) is unique comonoid on \(A \times B\).
- In \(\text{Rel}\), the product comonoid of groups \(G, H\) is comonoid of \(G \times H\) with multiplication \((g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)\).
- In \(\text{FHilb}\), the product of comonoids on \(H, K\) that copy bases \(\{d_i\}\) and \(\{e_j\}\) is the comonoid copying basis \(\{d_i \otimes e_j\}\) of \(H \otimes K\).
Product of monoids

Can combine two (co)monoids to single one using braiding:

If braiding is symmetry: categorical product.
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Dagger

Monoidal dagger category has duality between monoids and comonoids: \((A, d, e)\) is a comonoid if and only if \((A, d^\dagger, e^\dagger)\) is a monoid.

Example: In \(\text{Rel}\): comultiplication \(g \sim (h, h^{-1}g)\) for group \(G\) turns into multiplication \((g, h) \sim gh\).
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Closure

Morphisms transform *input* into *output*. But sometimes want to transform morphisms into morphisms.
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Morphisms transform *input* into *output*. But sometimes want to transform morphisms into morphisms. Can handle this using names and conames. E.g.:

\[ \text{FHilb}(H, K) = \{ H \xrightarrow{f} K \mid f \text{ linear} \} \]

is vector space with pointwise operations \((f + g)(x) = f(x) + g(x)\), Hilbert space with *trace inner product* \(\langle f \mid g \rangle = \text{Tr}(f^\dagger \circ g)\).

To transform morphisms, encode them as vectors in function spaces.
Matrices

One of most important features of matrices: they can be multiplied. In other words, linear maps \( \mathbb{C}^n \to \mathbb{C}^n \) can be composed. Using closure, can internalize this: the vector space \( \mathbb{M}_n \) of matrices is a monoid that lives in the same category as \( \mathbb{C}^n \).
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More generally, if an object $A$ in a monoidal category has a dual $A^*$, then operators $A \xrightarrow{f} A$ correspond bijectively to states $I \xrightarrow{\begin{bmatrix} f \end{bmatrix}} A^* \otimes A$. Composition $A \xrightarrow{g \circ f} A$ of operators transfers to states $I \xrightarrow{\begin{bmatrix} g \circ f \end{bmatrix}} A^* \otimes A$:

So $A^* \otimes A$ canonically becomes monoid.
Pair of pants

If $A \vdash A^*$ in monoidal category, then $A^* \otimes A$ is a monoid:

\[
\begin{array}{c}
A \\
\downarrow \quad \downarrow \\
A & A \\
\downarrow \quad \downarrow \\
\end{array}
\]

Proof.

\[
\begin{array}{c}
A \\
\downarrow \\
A & A \\
\end{array}
\]
Pair of pants

If $A \rightarrow A^*$ in monoidal category, then $A^* \otimes A$ is a monoid:

Proof.
Matrix algebras

Example: pair of pants on $\mathbb{C}^n$ in $\text{FHilb}$ is the algebra $\mathbb{M}_n$ of $n$-by-$n$ matrices under matrix multiplication.
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Proof: Fix basis $\{|i\rangle\}$ for $A = \mathbb{C}^n$, so $A^* \otimes A$ has basis $\{|j\rangle \otimes |i\rangle\}$.
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Define map $A^* \otimes A \rightarrow \mathbb{M}_n$ by mapping $\langle j \rangle \otimes \langle i \rangle$ to the matrix $e_{ij}$ with a single entry 1 on row $i$ and column $j$ and zeroes elsewhere.
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This bijection respects multiplication:

$$
\begin{align*}
\begin{array}{c}
\mathcal{P} \\
\hline
i & j \\
\hline
k & l
\end{array}
\end{align*}
= \begin{bmatrix} \langle i | \otimes |l\rangle & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix}
\quad \mapsto \quad 
\begin{bmatrix} e_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix} = e_{ij}e_{kl}
$$
Pair of pants are universal

Cayley: any group $G$ is a subgroup of a symmetric one.
Pair of pants are universal

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*Symmetric group $\text{Sym}(A)$*: bijections $A \to A$ under composition. Embedding $R: G \to \text{Sym}(G)$ is regular representation $g \mapsto R_g$.

$$R_g(h) = g \cdot h$$
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Already works for monoids: any $M$ is submonoid of $\text{Set}(M, M)$. 
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Already works for monoids: any $M$ is submonoid of $\text{Set}(M, M)$. Closure: instead of injective homomorphism $M \xrightarrow{R} \text{Set}(M, M)$, consider relation $M \to M^* \times M$ (latter with pair of pants).
Pair of pants are universal

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\[ R_g(h) = g \cdot h \]

Already works for monoids: any $M$ is submonoid of $\text{Set}(M, M)$. Closure: instead of injective homomorphism $M \xrightarrow{R} \text{Set}(M, M)$, consider relation $M \to M^* \times M$ (latter with pair of pants).

Abstract embedding of $(M, \triangleleft, \bullet)$ into $M \sqin M^*$:
Cayley’s theorem

Any monoid \( (A, \cdot, e) \) in a monoidal category with \( A \rightarrow A^* \) has monoid homomorphism to \( (A^* \otimes A, \cdot, e) \) with right inverse.

\[
\begin{array}{c}
R \\
\end{array}
\]

Proof.

\[ R \] preserves units:

\[ R \]

\[ \]

\[ R \] preserves multiplication:

\[ R \]

\[ \]

Finally, \( R \) has a right inverse.

\[ \]

\[ \]

\[ \]

\[ \]
Cayley’s theorem

Any monoid \((A, \cdot, e)\) in a monoidal category with \(A \dashv A^*\) has monoid homomorphism to \((A^* \otimes A, \land, \lrcorner)\) with right inverse.

\[
R = \quad = \quad
\]

**Proof.** \(R\) preserves units:

\[
R = \quad = \quad
\]
Cayley’s theorem

Any monoid \((A, \cdot, e)\) in a monoidal category with \(A \vdash A^*\) has monoid homomorphism to \((A^* \otimes A, \wedge, \vee)\) with right inverse.

\[
\begin{array}{c}
R \\
\end{array} = \begin{array}{c}
\text{Diagram}
\end{array}
\]

**Proof.** \(R\) preserves units:

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\begin{array}{c}
R \\
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\(R\) preserves multiplication:

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\begin{array}{c}
R \\
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\text{Diagram}
\end{array} = \begin{array}{c}
R \\
\end{array} \begin{array}{c}
R
\end{array}
\]

Finally, \(R\) has a right inverse \(\phi\).
Uniform deleting

Counit \(A \rightarrow I\) tells us we can ‘delete’ \(A\) if we want to.
What does it mean to have deletion \textit{systematically} on every object?

A monoidal category has uniform deleting if there is a natural transformation
\(A \rightarrow I\) with \(e_I = \text{id}_I\), such that:

\[A \otimes B \rightarrow I \otimes I \]

Uniform deleting possible if and only if \(I\) is terminal.

Proof.

Uniform deleting gives a morphism \(A \rightarrow I\) for each object \(A\).
Naturality and \(e_I = \text{id}_I\) then show any morphism \(A \rightarrow I\) equals \(e_A\).
Conversely, if \(I\) is terminal, choose \(e_A : A \rightarrow I\) uniquely.
Uniform deleting

Counit $A \xrightarrow{e} I$ tells us we can ‘delete’ $A$ if we want to. What does it mean to have deletion \textit{systematically} on every object?

A monoidal category has \textbf{uniform deleting} if there is a natural transformation $A \xrightarrow{e_A} I$ with $e_I = \text{id}_I$, such that:

\[
\begin{array}{ccc}
A \otimes B & \\
\downarrow^{e_A \otimes e_B} & \searrow^{e_{A \otimes B}} \\
I \otimes I & \xrightarrow{\lambda_I} & I
\end{array}
\]

Uniform deleting possible if and only if $I$ is terminal.

\textbf{Proof.}

Uniform deleting gives a morphism $A \xrightarrow{e_A} I$ for each object $A$. Naturality and $e_I = \text{id}_I$ then show any morphism $A \xrightarrow{f} I$ equals $e_A$.

Conversely, if $I$ is terminal, choose $e_A: A \xrightarrow{e_A} I$ uniquely.
Uniform deleting

Counit $A \xrightarrow{e} I$ tells us we can ‘delete’ $A$ if we want to. What does it mean to have deletion *systematically* on every object?

A monoidal category has **uniform deleting** if there is a natural transformation $A \xrightarrow{e_A} I$ with $e_I = \text{id}_I$, such that:

![Diagram]

Uniform deleting possible if and only if $I$ is terminal.
Uniform deleting

Counit $A \xrightarrow{e} I$ tells us we can ‘delete’ $A$ if we want to. What does it mean to have deletion systematically on every object?

A monoidal category has uniform deleting if there is a natural transformation $A \xrightarrow{e_A} I$ with $e_I = \text{id}_I$, such that:

\[
\begin{align*}
A \otimes B & \xrightarrow{e_A \otimes e_B} I \otimes I \\
& \xrightarrow{e_{A \otimes B}} I \\
& \xleftarrow{\lambda_I} I
\end{align*}
\]

Uniform deleting possible if and only if $I$ is terminal.

**Proof.** Uniform deleting gives a morphism $A \xrightarrow{e_A} I$ for each object $A$. Naturality and $e_I = \text{id}_I$ then show any morphism $A \xrightarrow{f} I$ equals $e_A$. Conversely, if $I$ is terminal, choose $e_A : A \rightarrow I$ uniquely. \qed
No-deleting theorem

A preorder is a category that has at most one morphism \( A \to B \) for any pair of objects \( A, B \).

Preorders are degenerate, with only process of each type.
No-deleting theorem

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**Theorem:** if a monoidal category with duals has uniform deleting, then it is a preorder.
No-deleting theorem

A preorder is a category that has at most one morphism $A \to B$ for any pair of objects $A, B$.

Preorders are degenerate, with only process of each type.

**Theorem:** if a monoidal category with duals has uniform deleting, then it is a preorder.

**Proof.** Let $A \xrightarrow{f,g} B$ be morphisms. Naturality of $e$ gives:

$$
\begin{align*}
A \otimes B^* &\xrightarrow{e_{A \otimes B^*}} I \\
I &\xrightarrow{e_I = \text{id}_I} I
\end{align*}
$$

So $\nabla f = e_{A \otimes B^*}$, and similarly $\nabla g = e_{A \otimes B^*}$. Hence $f = g$. 

\[\square\]
Uniform copying

Question: what does it mean to copy objects *systematically*?
Answer: copying must respect composition, tensor products.
Uniform copying

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Answer: copying must respect composition, tensor products.

A braided monoidal category has uniform copying if there is a natural transformation $A \xrightarrow{d_A} A \otimes A$ with $d_I = \rho_I$, satisfying cocommutativity and coassociativity, and:

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
d_A & & d_B \\
\downarrow & & \downarrow \\
A & \rightarrow & B \\
\end{array}
= 
\begin{array}{ccc}
A & & B \\
\downarrow & & \downarrow \\
d_A \otimes B & & d_B \\
\downarrow & & \downarrow \\
A & & B \\
\end{array}
$$
Uniform copying

Question: what does it mean to *copy* objects *systematically*?
Answer: copying must respect composition, tensor products.

A braided monoidal category has **uniform copying** if there is a natural transformation $\rho_I : I \to I$, satisfying cocommutativity and coassociativity, and:

$$\rho_I = I$$

Naturality and $\rho_I = \rho_I$ look like this for arbitrary $A \xrightarrow{f} B$:

$$\xrightarrow{\rho_I}$$
Copying states

Example: **Set** has uniform copying maps $a \mapsto (a, a)$:

\[
d_1(\bullet) = (\bullet, \bullet) = \rho_1(\bullet)
\]

both maps $A \times B \to A \times B \times A \times B$ are $(a, b) \mapsto (a, b, a, b)$
Copying states

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In a braided monoidal category, a state $I \xrightarrow{u} A$ is **copyable** with respect to a map $A \xrightarrow{d_A} A \otimes A$ when:

$$d_A \downarrow u = u \downarrow u$$
Copying states

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d_1(\bullet) = (\bullet, \bullet) = \rho_1(\bullet)
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both maps \( A \times B \to A \times B \times A \times B \) are \( (a, b) \mapsto (a, b, a, b) \)

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\[
d_A = u = u \otimes u
\]

In braided monoidal category with uniform copying, any state is copyable.
Copying states

Example: **Set** has uniform copying maps \( a \mapsto (a, a) \):
\[
d_1(\bullet) = (\bullet, \bullet) = \rho_1(\bullet)
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In a braided monoidal category, a state \( I \xrightarrow{u} A \) is **copyable** with respect to a map \( A \xrightarrow{d_A} A \otimes A \) when:

\[
\begin{array}{c}
\text{d}_A \\
\end{array}
= \\
\begin{array}{c}
\text{u} \\
\text{u} \\
\end{array}
\]

In braided monoidal category with uniform copying, any state is copyable.

**Proof.** If there is uniform copying, then, by naturality of the copying maps, we have \( d_A \circ u = (u \otimes u) \circ \rho_I \) for each state \( I \xrightarrow{u} A \). \( \square \)
Duals vs copying

If a braided monoidal category with duals has uniform copying:

\[ A^* \otimes A \cong A \otimes A^* \]

Proof.
First, consider the following equality (\( \ast \)):

\[ A^* \otimes A \cong A \otimes A^* \]

\[ d_A \otimes d_A \cong d_A \otimes d_A \]

Then:

\[ A^* \otimes A \otimes A^* \otimes A \cong A^* \otimes A \otimes A^* \otimes A \]

\[ A^* \otimes A \otimes A^* \otimes A \cong A^* \otimes A \otimes A^* \otimes A \]

\[ \quad \]

\[ A^* \otimes A \cong A \otimes A^* \]
Duals vs copying

If a braided monoidal category with duals has uniform copying:

$$A^* \otimes A \otimes A^* \otimes A = A^* \otimes A$$

**Proof.** First, consider the following equality (*):
Duals vs copying

If a braided monoidal category with duals has uniform copying:

\[
\begin{array}{c}
A^* \quad A \\
\quad \bigcirc \\
A \quad A^* \\
\quad \bigcirc
\end{array}
= 
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A
\end{array}

Proof. First, consider the following equality (*):

\[
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A
\end{array}
= 
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A
\end{array}
\begin{array}{c}
\text{id}_I
\end{array}
= 
\begin{array}{c}
\text{id}_{A^* \otimes A}
\end{array}
= 
\begin{array}{c}
\text{id}_{A^*}
\end{array}
\begin{array}{c}
\text{id}_A
\end{array}

Then:

\[
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A
\end{array}
= 
\begin{array}{c}
\text{id}_{A^*}
\end{array}
\begin{array}{c}
\text{id}_A
\end{array}
= 
\begin{array}{c}
\text{id}_{A^*}
\end{array}
\begin{array}{c}
\text{id}_A
\end{array}
= 
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A^* \\
\quad \bigcirc
\end{array}
\begin{array}{c}
A
\end{array}
\begin{array}{c}
\text{(*)}
\end{array}
Duals vs copying

In a braided monoidal category with duals and uniform copying:

\[ A^- \otimes A^{-} = A \otimes A \]

Proof.
Duals vs copying

In a braided monoidal category with duals and uniform copying:

\[
\begin{align*}
A & \quad A & = & \quad A & \quad A \\
\quad & \quad & = & \quad & \\
A & \quad A & = & \quad A & \quad A
\end{align*}
\]

Proof.

\[
\begin{align*}
A & \quad A & = & \quad \quad & = & \quad & \quad \quad & = & \quad & \quad \quad & = & \quad A & \quad A \\
A & \quad A & & & & & & & & & & \quad & \quad A & \quad A
\end{align*}
\]
No-cloning theorem

If a braided monoidal category with duals has uniform copying, every endomorphism is a multiple of the identity, $f = \text{Tr}(f) \cdot \text{id}$:
No-cloning theorem

If a braided monoidal category with duals has uniform copying, every endomorphism is a multiple of the identity, \( f = \text{Tr}(f) \cdot \text{id} \):

\[
\begin{align*}
\begin{array}{c}
\text{Proof.} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Proof.} \\
\end{array}
\end{align*}
\]
Products

The following are equivalent for a symmetric monoidal category:

- tensor products are products and the tensor unit is terminal
- it has uniform copying and deleting, satisfying counitality
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**Proof.** If cartesian, unique $A \xrightarrow{e_A} I$ and $d_A = \begin{pmatrix} \text{id}_A \\ \text{id}_A \end{pmatrix}$ provide uniform copying and deleting.
Products

The following are equivalent for a symmetric monoidal category:

- tensor products are products and the tensor unit is terminal
- it has uniform copying and deleting, satisfying counitality

**Proof.** If cartesian, unique $A \xrightarrow{e_A} I$ and $d_A = \begin{pmatrix} \text{id}_A \\ \text{id}_A \end{pmatrix}$ provide uniform copying and deleting.

For converse, need to prove $A \otimes B$ is product of $A, B$. For $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$, define

\[
\begin{pmatrix} f \\ g \end{pmatrix} = (f \otimes g) \circ d
\]

\[
p_A = \rho_A \circ (\text{id}_A \otimes e_B): A \otimes B \rightarrow A
\]

\[
p_B = \lambda_B \circ (e_A \otimes \text{id}_B): A \otimes B \rightarrow B
\]
Proof. Suppose $C \xrightarrow{m} A \otimes B$ satisfies $p_A \circ m = f$ and $p_B \circ m = g$. 

Hence mediating morphisms, if they exist, are unique. Finally, we show the universal morphism has the right properties: $p_B \circ (f \circ g) = e_A \circ d_C \circ g = e_C \circ d_C \circ g = g$.

A similar result holds for $g$. 

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Proof. Suppose $C \xrightarrow{m} A \otimes B$ satisfies $p_A \circ m = f$ and $p_B \circ m = g$. Then:

$$(f \circ g) = \begin{array}{c} f \\ \hline \end{array} \begin{array}{c} g \\ \hline \end{array} = \begin{array}{c} e_B \\ \hline \end{array} \begin{array}{c} e_A \\ \hline \end{array} \begin{array}{c} m \\ \hline \end{array} = \begin{array}{c} e_B \\ \hline \end{array} \begin{array}{c} e_A \\ \hline \end{array} \begin{array}{c} d_{A \otimes B} \\ \hline \end{array} = \begin{array}{c} d_A \\ \hline \end{array} \begin{array}{c} d_B \\ \hline \end{array} = m$$

Hence mediating morphisms, if they exist, are unique.
Proof. Suppose $C \xrightarrow{m} A \otimes B$ satisfies $p_A \circ m = f$ and $p_B \circ m = g$. Then:

$$
(f \ g) = \begin{array}{c}
\begin{array}{c}
| \hfill f \hfill |
\hfill g \hfill |
\hfill d_C \hfill |
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| e_B \hfill |
\hfill e_A \hfill |
\hfill m \hfill |
\end{array}
\hfill m \hfill |
\hfill d_C \hfill |
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| e_B \hfill |
\hfill e_A \hfill |
\hfill d_{A \otimes B} \hfill |
\end{array}
\hfill m \hfill |
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| e_B \hfill |
\hfill e_A \hfill |
\hfill d_A \hfill |
\end{array}
\hfill d_B \hfill |
\end{array}
\end{array} = m
$$

Hence mediating morphisms, if they exist, are unique.

Finally, we show the universal morphism has the right properties:

$$
p_B \circ (f \ g) = \begin{array}{c}
\begin{array}{c}
| e_A \hfill |
\hfill f \hfill |
\hfill g \hfill |
\hfill d_C \hfill |
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| e_C \hfill |
\hfill g \hfill |
\hfill d_C \hfill |
\end{array}
\end{array}
\end{array} = g
$$

A similar result holds for $g$. 

\(\square\)
Summary

- Monoids: multiplication on states
- Comonoids: ‘copying’ of states
- Closure: operators form monoids
- Cloning: no-cloning and no-deleting
- Products: characterize when tensor product is product

Next week: interaction between monoids and comonoids