Categories and Quantum Informatics exercise sheet 5: Monoids and comonoids

Exercise 4.1. Let (A, d, e) be a comonoid in a monoidal category. Show that a comonoid homomorphism $I \xrightarrow{a} A$ is a copyable state. Conversely, show that if a state $I \xrightarrow{a} A$ is copyable and satisfies $e \circ a = \mathrm{id}_I$, then it is a comonoid homomorphism.

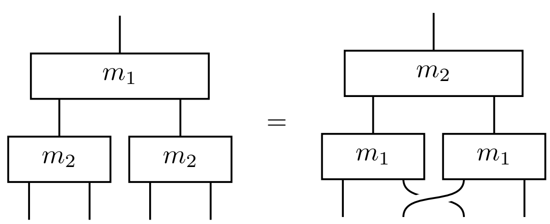
Exercise 4.2. This exercise is about *property* versus *structure*; the latter is something you have to choose, the former is something that exists uniquely (if at all).

(a) Show that if a monoid (A, m, u) in a monoidal category has a map $I \xrightarrow{u'} A$ satisfying $m \circ (\mathrm{id}_A \otimes u') = \rho_A$ and $\lambda_A = m \circ (u' \otimes \mathrm{id}_A)$, then $u' = u$. Conclude that unitality is a property.

(b) Show that in categories with binary products and a terminal object, every object has a unique comonoid structure under the monoidal structure induced by the categorical product.

(c) If $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal category, denote by $\mathbf{cMon}(\mathbf{C})$ the category of commutative monoids in $\mathbf{C}$ with monoid homomorphisms as morphisms. Show that the forgetful functor $\mathbf{cMon}(\mathbf{C}) \longrightarrow \mathbf{C}$ is an isomorphism of categories if and only if $\otimes$ is a coproduct and I is an initial object.

Exercise 4.3. This exercise is about the *Eckmann–Hilton argument*, concerning interacting monoid structures on a single object in a braided monoidal category. Suppose you have morphisms $A \otimes A \xrightarrow{m_1, m_2} A$ and $I \xrightarrow{u_1, u_2} A$, such that (A, m_1, u_1) and (A, m_2, u_2) are both monoids, and the following diagram commutes:



(a) Show that $u_1 = u_2$.

(b) Show that $m_1 = m_2$.

(c) Show that m_1 is commutative.