Categories and Quantum Informatics exercise sheet 5: 
Monoids and comonoids

Exercise 4.1. Let $(A, d, e)$ be a comonoid in a monoidal category. Show that a comonoid homomorphism $I \otimes A$ is a copyable state. Conversely, show that if a state $I \otimes A$ is copyable and satisfies $e \circ a = id_I$, then it is a comonoid homomorphism.

Exercise 4.2. This exercise is about property versus structure; the latter is something you have to choose, the former is something that exists uniquely (if at all).

(a) Show that if a monoid $(A, m, u)$ in a monoidal category has a map $I \rightarrow A$ satisfying $m \circ (id_A \otimes u') = \rho_A$ and $\lambda_A = m \circ (u' \otimes id_A)$, then $u' = u$. Conclude that unitality is a property.

(b) Show that in categories with binary products and a terminal object, every object has a unique comonoid structure under the monoidal structure induced by the categorical product.

(c) If $(C, \otimes, I)$ is a symmetric monoidal category, denote by $cMon(C)$ the category of commutative monoids in $C$ with monoid homomorphisms as morphisms. Show that the forgetful functor $cMon(C) \rightarrow C$ is an isomorphism of categories if and only if $\otimes$ is a coproduct and $I$ is an initial object.

Exercise 4.3. This exercise is about the Eckmann–Hilton argument, concerning interacting monoid structures on a single object in a braided monoidal category. Suppose you have morphisms $A \otimes A \rightarrow A$ and $I \rightarrow A$, such that $(A, m_1, u_1)$ and $(A, m_2, u_2)$ are both monoids, and the following diagram commutes:

\[
\begin{array}{ccc}
    m_1 & \rightarrow & m_2 \\
    \downarrow & & \downarrow \\
    m_2 & & m_2
\end{array}
\]

= \[
\begin{array}{ccc}
    m_1 & \rightarrow & m_1 \\
    \downarrow & & \downarrow \\
    m_1 & & m_1
\end{array}
\]

(a) Show that $u_1 = u_2$.

(b) Show that $m_1 = m_2$.

(c) Show that $m_1$ is commutative.