Categories and Quantum Informatics exercise sheet 5: Monoids and comonoids

Exercise 4.1. The comonoid structure on I is given by $(I, \lambda_I^{-1}, \mathrm{id}_I)$. The definition of copyability and the first part of the definition of comonoid homomorphism are both described by the same equation in this case, namely:

$$d \circ a = (a \otimes a) \circ \lambda_I^{-1}$$

This means that a state a is copyable iff a satisfies the first equation in the definition of comonoid homomorphism. Note that in general, a copyable state a does not satisfy the other condition, namely deletion. A counter example is taking a zero state.

Exercise 4.2. (a) The graphical proof for this part is very simple (simply plug in both u and u' into m), but we present a symbolic one for comparisson.
Observe that the following equation holds because of naturality of ρ:

$$\rho_A \circ (u \otimes id_I) = u \circ \rho_I$$

Since $\lambda_I = \rho_I$, we have:

$$\rho_A \circ (u \otimes id_I) = u \circ \rho_I = u \circ \lambda_I$$

Using the same argument, but for λ and u' we get:

$$\lambda_A \circ (id_I \otimes u') = u' \circ \lambda_I = u' \circ \rho_I$$

We have:

$$\begin{aligned}
& m \circ (id_A \otimes u') = \rho_A \\
\Longrightarrow\quad & m \circ (id_A \otimes u') \circ (u \otimes id_I) = \rho_A \circ (u \otimes id_I) && \text{(compose on right)} \\
\Longrightarrow\quad & m \circ (id_A \otimes u') \circ (u \otimes id_I) = u \circ \lambda_I && \text{(above equation)} \\
\Longrightarrow\quad & m \circ (u \otimes u') = u \circ \lambda_I && \text{(interchange law)} \\
\Longrightarrow\quad & m \circ (u \otimes id_A) \circ (id_I \otimes u') = u \circ \lambda_I && \text{(interchange law)} \\
\Longrightarrow\quad & \lambda_A \circ (id_I \otimes u') = u \circ \lambda_I && \text{(monoid axiom)} \\
\Longrightarrow\quad & u' \circ \lambda_I = u \circ \lambda_I && \text{(above equation)} \\
\Longrightarrow\quad & u' = u && (\lambda_I \text{ is invertible})
\end{aligned}$$

Note, that we have used only one of the equations for u'.

(b) We will write the product of $f : X \longrightarrow A$ and $g : X \longrightarrow B$ as $\langle f, g\rangle : X \longrightarrow A \times B$ and the associated projections will be written as $\pi_1^{A\times B}$ and $\pi_2^{A\times B}$.
Recall, that

$$\pi_1^{A\times B} \circ \langle f, g\rangle = f$$
$$\pi_2^{A\times B} \circ \langle f, g\rangle = g$$

$$\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$$

First, we need to express the monoidal structure induced by the product. It is given in the following way:
For objects, A and B

$$A \otimes B := A \times B$$

For morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow D$

$$f \otimes g := \langle f \circ \pi_1^{A\times C}, g \circ \pi_2^{A\times C}\rangle$$

The monoidal unit I is the terminal object 1 of the category. Then,

$$\lambda_A := \pi_2^{1\times A}$$

$$\rho_A := \pi_1^{A\times 1}$$

$$\alpha_{A,B,C} := \langle \pi_1^{A\times B} \circ \pi_1^{(A\times B)\times C}, \langle \pi_2^{A\times B} \circ \pi_1^{(A\times B)\times C}, \pi_2^{(A\times B)\times C}\rangle\rangle$$

Next, we need to show that every object in the category has a comonoid structure. Let A be an arbitrary object. We can assign it a comonoid structure (A, d, e) by defining:

$$d := \langle id_A, id_A\rangle : A \longrightarrow A \times A$$

$$e := 1_A : A \longrightarrow 1$$

where 1_A is the unique morphism going from A to the terminal object 1. We have to verify that the axioms for a comonoid are satisfied.

$$\begin{aligned}
\rho_A \circ (id_A \otimes e) \circ d &= \pi_1^{A\times 1} \circ \langle id_A \circ \pi_1^{A\times A}, 1_A \circ \pi_2^{A\times A}\rangle \circ \langle id_A, id_A\rangle \\
&= id_A \circ \pi_1^{A\times A} \circ \langle id_A, id_A\rangle \\
&= id_A
\end{aligned}$$

as required. Next,

$$\begin{aligned}
\lambda_A \circ (e \otimes id_A) \circ d &= \pi_2^{1\times A} \circ \langle 1_A \circ \pi_1^{A\times A}, id_A \circ \pi_2^{A\times A}\rangle \circ \langle id_A, id_A\rangle \\
&= id_A \circ \pi_2^{A\times A} \circ \langle id_A, id_A\rangle \\
&= id_A
\end{aligned}$$

as required. Next, we show coassociativity:

$$\begin{aligned}
\alpha_{A,A,A} \circ (d \otimes id_A) \circ d &= \alpha_{A,A,A} \circ \langle d \circ \pi_1^{A\times A}, id_A \circ \pi_2^{A\times A}\rangle \circ d \\
&= \alpha_{A,A,A} \circ \langle d \circ \pi_1^{A\times A} \circ d, \pi_2^{A\times A} \circ d\rangle \\
&= \alpha_{A,A,A} \circ \langle d \circ id_A, id_A\rangle \\
&= \langle \pi_1^{A\times A} \circ \pi_1^{(A\times A)\times A}, \langle \pi_2^{A\times A} \circ \pi_1^{(A\times A)\times A}, \pi_2^{(A\times A)\times A}\rangle\rangle \circ \langle d, id_A\rangle \\
&= \langle \pi_1^{A\times A} \circ \pi_1^{(A\times A)\times A} \circ \langle d, id_A\rangle, \langle \pi_2^{A\times A} \circ \pi_1^{(A\times A)\times A}, \pi_2^{(A\times A)\times A}\rangle \circ \langle d, id_A\rangle\rangle \\
&= \langle \pi_1^{A\times A} \circ d, \langle \pi_2^{A\times A} \circ \pi_1^{(A\times A)\times A} \circ \langle d, id_A\rangle, \pi_2^{(A\times A)\times A} \circ \langle d, id_A\rangle\rangle\rangle \\
&= \langle id_A, \langle \pi_2^{A\times A} \circ d, id_A\rangle\rangle \\
&= \langle id_A, \langle id_A, id_A\rangle\rangle
\end{aligned}$$

Also,

$$\begin{aligned}(id_A \otimes d) \circ d &= \langle id_A \circ \pi_1^{A\times A}, d \circ \pi_2^{A\times A}\rangle \circ d \\ &= \langle \pi_1^{A\times A} \circ d, d \circ \pi_2^{A\times A} \circ d\rangle \\ &= \langle id_A, d \circ id_A\rangle \\ &= \langle id_A, d\rangle \\ &= \langle id_A, \langle id_A, id_A\rangle\rangle\end{aligned}$$

Therefore, coassociativity holds and (A, d, e) is indeed a comonoid.
Next, we have to show that the construction is unique. That is, for any other comonoid (A, d', e') that $d = d'$ and $e = e'$.
Since 1 is a terminal object, then it must be the case that $e = e' : A \longrightarrow 1$. From counitlaity of $(A, d', e' = e)$ we have:

$$\begin{aligned}id_A &= \rho_A \circ (id_A \otimes e) \circ d' \\ &= \pi_1^{A\times 1} \circ \langle id_A \circ \pi_1^{A\times A}, 1_A \circ \pi_2^{A\times A}\rangle \circ d' \\ &= id_A \circ \pi_1^{A\times A} \circ d' \\ &= \pi_1^{A\times A} \circ d'\end{aligned}$$

and also,

$$\begin{aligned}id_A &= \lambda_A \circ (e \otimes id_A) \circ d' \\ &= \pi_2^{1\times A} \circ \langle 1_A \circ \pi_1^{A\times A}, id_A \circ \pi_2^{A\times A}\rangle \circ d' \\ &= id_A \circ \pi_2^{A\times A} \circ d' \\ &= \pi_2^{A\times A} \circ d'\end{aligned}$$

Because of these two equalities and from the universal property of categorical products, it then follows that d' must be the unique morphism

$$d' = \langle id_A, id_A\rangle = d$$

which completes the proof.

(c) **RHS $\Rightarrow$ LHS:** Since $\otimes$ is a coproduct, we can simply use the dualized statement of (b) to conclude that every object A has a unique monoid structure (A, m_A, u_A).
First, note that the tensor product on two morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow D$ is given by:

$$f \otimes g := \left[i_1^{B\oplus D} \circ f, i_2^{B\oplus D} \circ g\right] : A \oplus C \longrightarrow B \oplus D$$

and braiding is given by:

$$\sigma_{A,B} := [i_2^{B\oplus A}, i_1^{B\oplus A}] : A \oplus B \longrightarrow B \oplus A$$

The monoidal structure on A is defined by:

$$m_A := [id_A, id_A] : A \oplus A \longrightarrow A$$

$$u_A := 1_A : I \longrightarrow A$$

where 1_A is unique morphism from the initial object to A. Also, recall that:

$$\begin{aligned}
[f,g] \circ i_1^{B\oplus D} &= f \\
[f,g] \circ i_2^{B\oplus D} &= g \\
h \circ [f,g] &= [h \circ f, h \circ g]
\end{aligned}$$

We show that every monoid (A, m_A, u_A) is commutative:

$$\begin{aligned}
m_A \circ \sigma_{A,A} &= [id_A, id_A] \circ [i_2^{B\oplus A}, i_1^{B\oplus A}] && \text{(definition)} \\
&= [[id_A, id_A] \circ i_2^{B\oplus A}, [id_A, id_A] \circ i_1^{B\oplus A}] && \text{(coproduct)} \\
&= [id_A, id_A] && \text{(coproduct)} \\
&= m_A && \text{(definition)}
\end{aligned}$$

We can definie an isomorphism $F : \mathbf{C} \longrightarrow \mathbf{cMon(C)}$ in the following way:

$$\begin{aligned}
F(A) &:= (A, m_A, u_A) \\
F(f) &:= f
\end{aligned}$$

It's clear that this functor is an isomorphism, if it is well-defined. We have already shown it is well-defined on objects. We just have to show that every morphism in $\mathbf{C}$ is a monoid homomorphism. Let $f : A \longrightarrow B$ be an arbitrary morphism. Now, consider the monoidal structures of the two objects (A, m_A, u_A), (B, m_B, u_B). We have:

$$\begin{gathered}
u_B = f \circ u_A \\
\iff \\
1_B = f \circ 1_A
\end{gathered}$$

which is clearly true, since $I = 1$ is an initial object.

$$\begin{aligned}
& f \circ m_A = m_B \circ (f \otimes f) \\
\iff & f \circ [id_A, id_A] = [id_B, id_B] \circ [i_1^{B\oplus B} \circ f, i_2^{B\oplus D} \circ f] \\
\iff & [f,f] = [[id_B, id_B] \circ i_1^{B\oplus B} \circ f, [id_B, id_B] \circ i_2^{B\oplus D} \circ f] \\
\iff & [f,f] = [id_B \circ f, id_B \circ f] \\
\iff & [f,f] = [f,f]
\end{aligned}$$

Therefore, f is a monoid homomorphism and thus F is an isomorphism. Showing that the functor F is a monoidal functor is straightforward with all of the definitions we have provided.

LHS $\Rightarrow$ RHS: cMon(C) is monoidally isomorphic to $\mathbf{C}$ therefore every object A has a unique monoid structure which we will denote as (A, m_A, u_A). Consider objects A, B, C and $A \otimes B$ and morphisms $f_1 : A \longrightarrow C$, $f_2 : B \longrightarrow C$. Since the categories are isomorphic, this implies that f_1 and f_2 are monoid homomorphisms.

First, we define morphisms $i_1^{A\otimes B} : A \longrightarrow A \otimes B$, $i_2^{A\otimes B} : B \longrightarrow A \otimes B$ given by:

$$i_1 := (id_A \otimes u_B) \circ \rho_A^{-1}$$

$$i_2 := (u_A \otimes id_B) \circ \lambda_B^{-1}$$

Define,

$$[f_1, f_2] := m_C \circ (f_1 \otimes f_2)$$

We claim that $([f_1, f_2], i_1, i_2)$ is the coproduct of the morphisms f_1 and f_2. First, we verify:

$$\begin{aligned}
[f_1, f_2] \circ i_1 &= m_C \circ (f_1 \otimes f_2) \circ (id_A \otimes u_B) \circ \rho_A^{-1} && \text{(definition)}\\
&= m_C \circ (f_1 \otimes (f_2 \circ u_B)) \circ \rho_A^{-1} && \text{(interchange)}\\
&= m_C \circ (f_1 \otimes u_C) \circ \rho_A^{-1} && \text{(monoid homomorphism)}\\
&= m_C \circ (id_C \otimes u_C) \circ (f_1 \otimes id_I) \circ \rho_A^{-1} && \text{(interchange)}\\
&= \rho_C \circ (f_1 \otimes id_I) \circ \rho_A^{-1} && \text{(monoid unitality for C)}\\
&= f_1 \circ \rho_A \circ \rho_A^{-1} && \text{(naturality of } \rho\text{)}\\
&= f_1
\end{aligned}$$

In a similar way, we can show that

$$[f_1, f_2] \circ i_2 = f_2$$

Finally, we have to show that the construction is universal. That is, if there exists a morphism $h : A \otimes B \longrightarrow C$ with $h \circ i_1 = f_1$ and $h \circ i_2 = f_2$ then $h = [f_1, f_2]$.
Consider:

$$\begin{aligned}
[f_1, f_2] &= m_C \circ (f_1 \otimes f_2) && \text{(definition)}\\
&= m_C \circ ((h \circ i_1) \otimes (h_2 \circ i_2)) && \text{(assumption)}\\
&= m_C \circ (h \otimes h) \circ (i_1 \otimes i_2) && \text{(interchange)}\\
&= h \circ m_{A\otimes B} \circ (i_1 \otimes i_2) && (h - \text{homomorphism})\\
&= h \circ (m_A \otimes m_B) \circ (id_A \circ \sigma_{B,A} \circ id_B) \circ\\
&\circ (id_A \otimes u_B \otimes u_A \otimes id_B) \circ (\rho_A^{-1} \otimes \lambda_B^{-1}) && \text{(def+interchange)}\\
&= h \circ (m_A \otimes m_B) \circ (id_A \circ u_A \circ u_B \circ id_B) \circ (\rho_A^{-1} \otimes \lambda_B^{-1}) && \text{(interchange)}\\
&= h \circ (\rho_A \otimes \lambda_B) \circ (\rho_A^{-1} \otimes \lambda_B^{-1}) && \text{(unitality} \times 2\text{)}\\
&= h && \text{(interchange)}
\end{aligned}$$

We have shown that coproducts exist for any pair of objects A and B. However, we still need to show that there is an initial object. The initial object is, of course, the tensor unit I. Consider an arbitrary object A. Since A is a monoid, then there must be a map $u_A : I \longrightarrow A$. Moreover, if there is another morphism $x : I \longrightarrow A$, then it must be a monoid homomorphism. Therefore,

$$u_A = x \circ u_I = x \circ id_I = x$$

since (I, λ_I, id_I) is the unique monoid on I.
Therefore, I is an initial object, which completes the proof.

Exercise 4.3. For the whole exercise, the graphical proof is very simple and straightforward. However, for comparisson, we show a symbolic solution instead.

(a) The trick is to plug in the state $(u_2 \otimes u_1 \otimes u_1 \otimes u_2)$.

$$\begin{aligned}
m_1 \circ (m_2 \otimes m_2) \circ (u_2 \otimes u_1 \otimes u_1 \otimes u_2) &= m_2 \circ (m_1 \otimes m_1) \circ (id_A \circ \sigma \circ id_A) \circ (u_2 \otimes u_1 \otimes u_1 \otimes u_2) \\
&\Longrightarrow \\
m_1 \circ (\lambda_A \circ (id_I \otimes u_1)) \otimes (\rho_A \circ (u_1 \otimes id_I)) &= m_2 \circ (m_1 \otimes m_1) \circ (u_2 \otimes u_1 \otimes u_1 \otimes u_2) \\
&\Longrightarrow \\
m_1 \circ ((u_1 \circ \lambda_I) \otimes (u_1 \circ \rho_I)) &= m_2 \circ ((\rho_A \circ (u_2 \otimes id_I)) \otimes (\lambda_A \circ (id_I \otimes u_2))) \\
&\Longrightarrow \\
m_1 \circ (u_1 \otimes u_1) \circ (\lambda_I \otimes \rho_I)) &= m_2 \circ ((u_2 \circ \rho_I) \otimes (u_2 \circ \lambda_I)) \\
&\Longrightarrow \\
\lambda_A \circ (id_I \otimes u_1) \circ (\lambda_I \otimes \rho_I)) &= m_2 \circ (u_2 \otimes u_2) \circ (\rho_I \otimes \lambda_I) \\
&\Longrightarrow \\
\lambda_A \circ (id_I \otimes u_1) &= m_2 \circ (u_2 \otimes u_2) \\
&\Longrightarrow \\
\lambda_A \circ (id_I \otimes u_1) &= \lambda_A \circ (id_I \otimes u_2) \\
&\Longrightarrow \\
u_1 \circ \lambda_I &= u_2 \circ \lambda_I \\
&\Longrightarrow \\
u_1 &= u_2
\end{aligned}$$

(b) From now on we will write $u := u_1 = u_2$.
Plugging in the map $(id_A \otimes u \otimes u \otimes id_A)$ to both sides of the equation yields the desired result.

$$\begin{aligned}
m_1 \circ (m_2 \otimes m_2) \circ (id_A \otimes u \otimes u \otimes id_A) &= m_2 \circ (m_1 \otimes m_1) \circ (id_A \circ \sigma \circ id_A) \circ (id_A \otimes u \otimes u \otimes id_A) \\
&\Longrightarrow \\
m_1 \circ (m_2 \otimes m_2) \circ (id_A \otimes u \otimes u \otimes id_A) &= m_2 \circ (m_1 \otimes m_1) \circ (id_A \otimes u \otimes u \otimes id_A) \\
&\Longrightarrow \\
m_1 \circ (\rho_A \otimes \lambda_A) &= m_2 \circ (\rho_A \otimes \lambda_A) \\
&\Longrightarrow \\
m_1 &= m_2
\end{aligned}$$

(c) We will write $m := m_1 = m_2$.
This time, the trick is to plug in the map $(u \otimes id_A \otimes id_A \otimes u)$ to both sides of the equation. We get:

$$\begin{aligned}
m \circ (m \otimes m) \circ (u \otimes id_A \otimes id_A \otimes u) &= m \circ (m \otimes m) \circ (id_A \circ \sigma \circ id_A) \circ (u \otimes id_A \otimes id_A \otimes u) \\
&\Longrightarrow \\
m \circ (\lambda_A \otimes \rho_A) &= m \circ (m \otimes m) \circ (u \otimes id_A \otimes id_A \otimes u) \circ (id_I \circ \sigma \circ id_I) \\
&\Longrightarrow \\
m \circ (\lambda_A \otimes \rho_A) &= m \circ (\lambda_A \otimes \rho_A) \circ (id_I \circ \sigma \circ id_I) \\
&\Longrightarrow \\
m \circ (\lambda_A \otimes \rho_A) &= m \circ \sigma \circ (\lambda_A \otimes \rho_A) \\
&\Longrightarrow \\
m &= m \circ \sigma
\end{aligned}$$